# Dynamics of Systems

# CTB 2300


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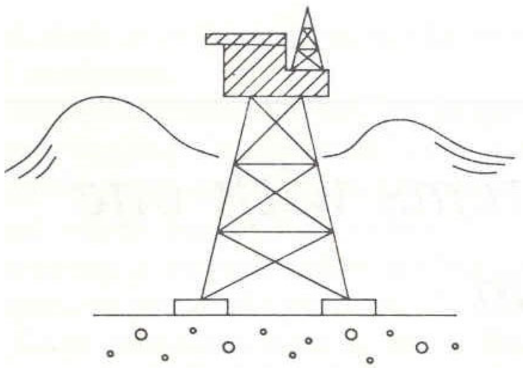
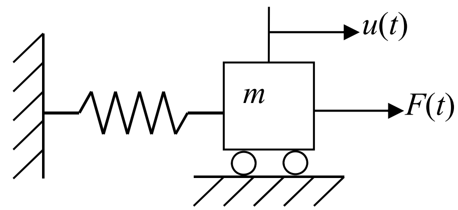
# Contents of Lecture 2

Free vibration of Single Degree of Freedom systems (SDOFs)

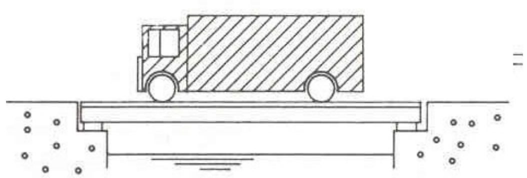
*examples of SDOF idealizations, translational and rotational SDOFs, two methods of formulation of the equation of motion, free vibration, the general solution of the homogeneous equation, the natural frequency, initial conditions, energy of vibration*

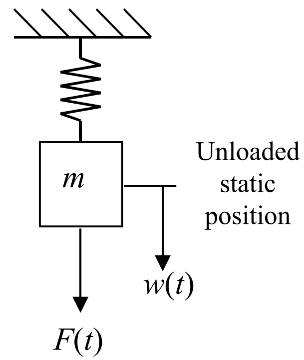
# Examples of SDoF Idealizations





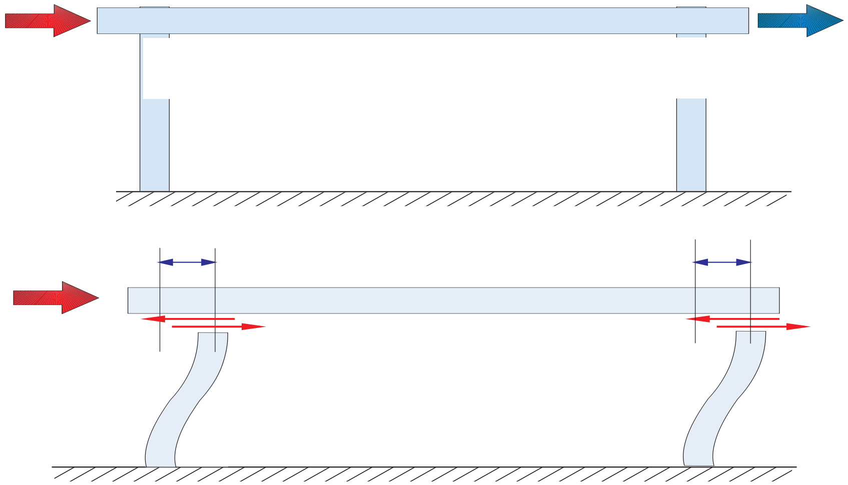

Idealization of bending motion of an offshore platform in waves



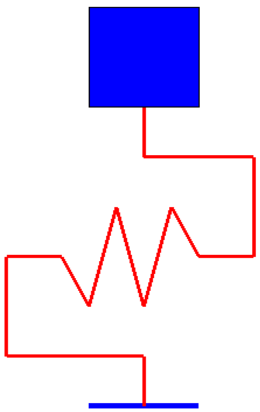



Idealization of vertical vibration of a lorry on a bridge

# Visualization of the free vibration of an SDOF



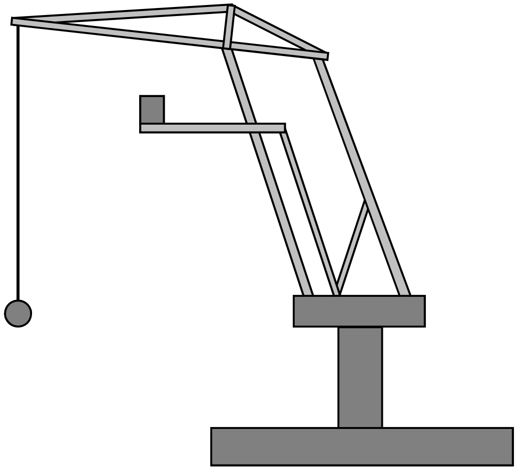
**Video**:
free vibration of
Japanese skyscrapers
after 2011 Earthquake

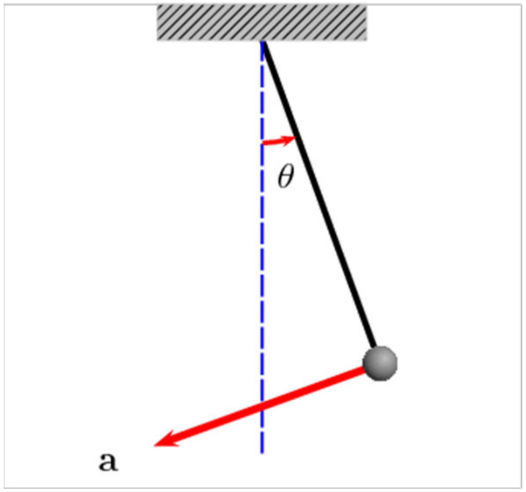


Please look at the file Lecture_2_MAPLE_2_1.

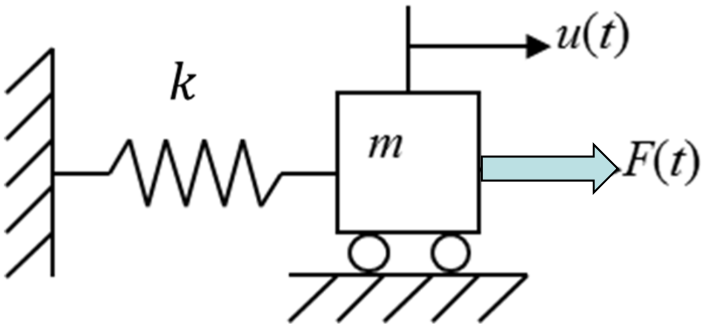
It is available on the Brightspace

# Visualization of the free vibration of an SDOF

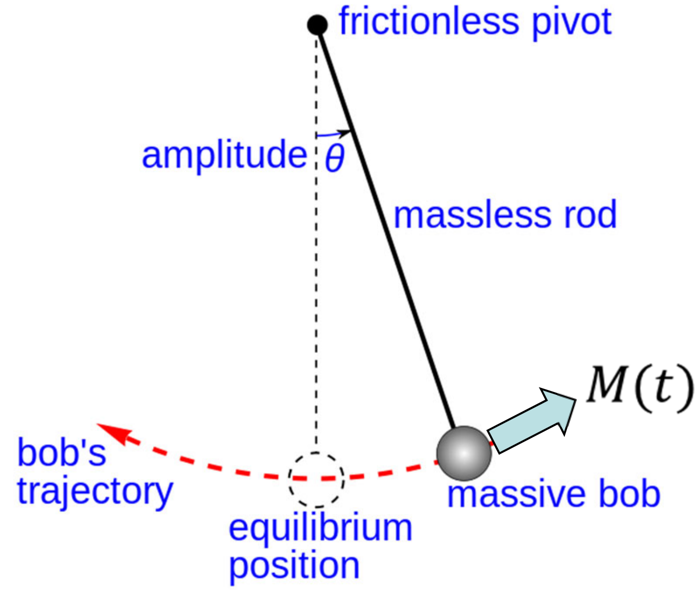




# Two mechanical SDoFs to consider

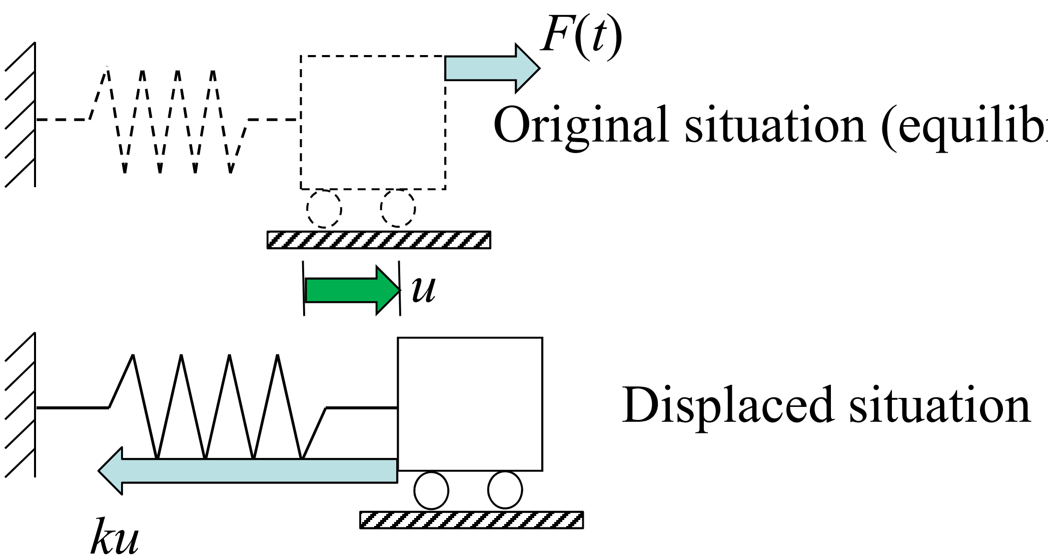



Translational oscillator:
an elastically constrained mass
in translational motion




Rotational oscillator (pendulum):
a rigidly constrained mass in orbital motion

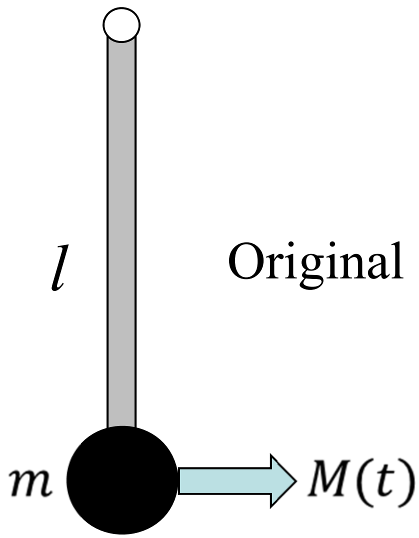
# Formulation of the equation of motion
# Method 1: The displacement method



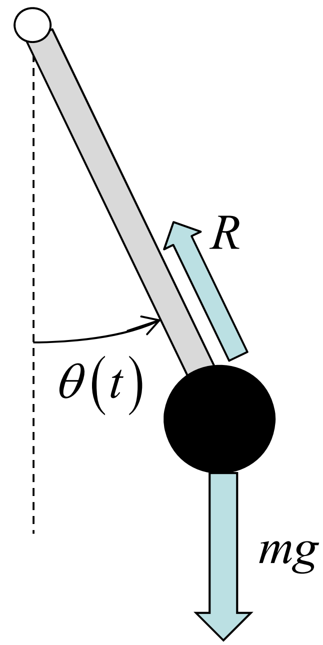

Equation of motion (dynamic equilibrium of the linear momentum)

$$m\frac{d^2u}{dt^2} = -ku + F(t)$$

# Formulation of the equation of motion
# Method 1: The displacement method




Original situation (equilibrium)




Displaced situation

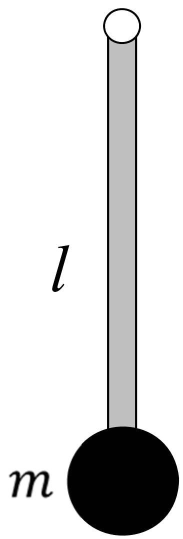
Equation of motion (dynamic equilibrium of the angular momentum)

$$ml^2 \frac{d^2\theta}{dt^2} = -mgl\sin(\theta) + M(t)$$

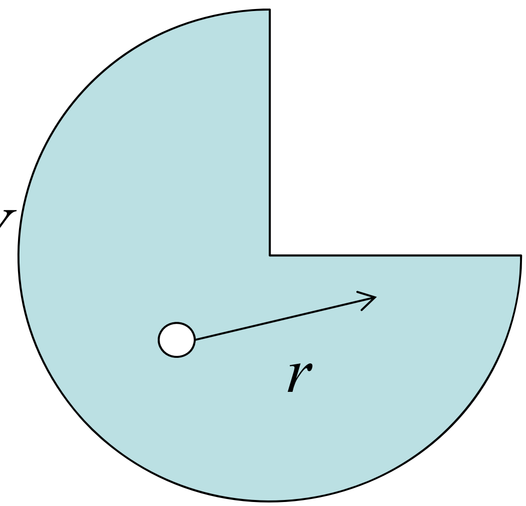
# Intermezzo: The mass moment of inertia

https://www.youtube.com/watch?v=CHQOctEvtTY

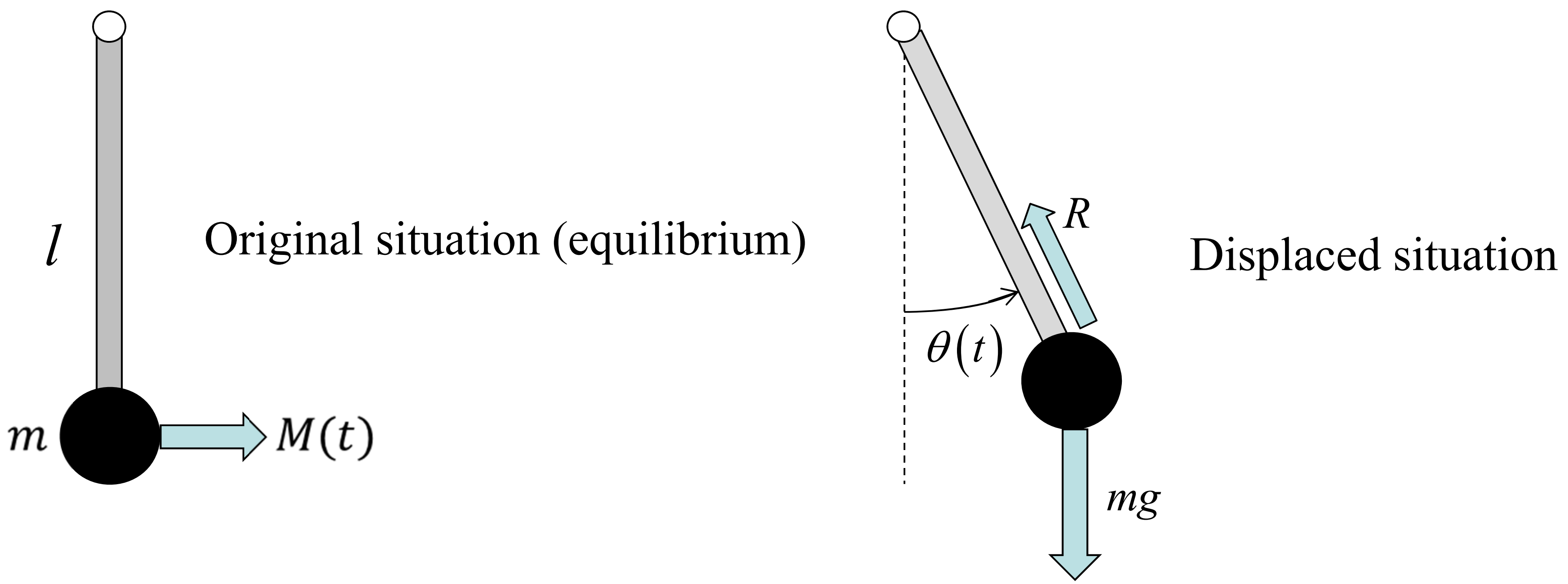
$$J = ml^2$$




$$J = \int_V \rho(\vec{r}) |\vec{r}|^2 \, dV$$



# Formulation of the equation of motion
## Method 1: The displacement method




For small angles: $$ml^2 \frac{d^2\theta}{dt^2} = -mgl\theta + M(t)$$

# Formulation of the equation of motion
# Method 2: The Lagrangian formalism

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = Q$$

$$\mathcal{L} = \mathcal{K} - \mathcal{P}$$

$\mathcal{L}$- The Lagrange Function

$\mathcal{K}, \mathcal{P}$ - Kinetic and Potential Energies

$q(t)$ is a generalized coordinate

$Q(t)$ is a generalized force



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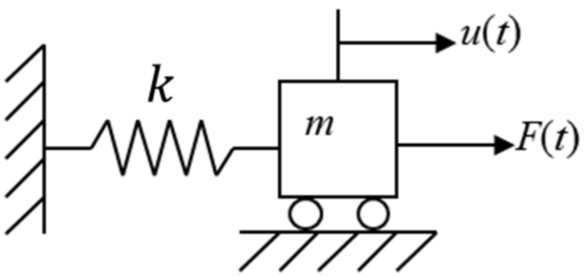
# Formulation of the equation of motion
## Method 2: The Lagrangian formalism



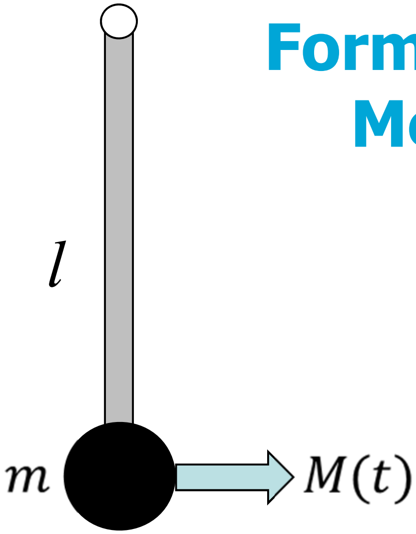

$$q(t) = u(t)$$
$$Q(t) = F(t)$$

$$\mathcal{K} = \frac{1}{2}m(\dot{u})^2, \quad \mathcal{P} = \frac{1}{2}ku^2$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{u}} = \frac{d}{dt}\frac{\partial}{\partial \dot{u}}\left(\frac{1}{2}m(\dot{u})^2 - \frac{1}{2}ku^2\right) = \frac{d}{dt}\frac{\partial}{\partial \dot{u}}\left(\frac{1}{2}m(\dot{u})^2\right) = \frac{d}{dt}(m\dot{u}) = m\ddot{u}$$

$$\frac{\partial \mathcal{L}}{\partial u} = \frac{\partial}{\partial u}\left(\frac{1}{2}m(\dot{u})^2 - \frac{1}{2}ku^2\right) = \frac{\partial}{\partial u}\left(-\frac{1}{2}ku^2\right) = -ku$$

Equation of motion: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{u}} - \frac{\partial \mathcal{L}}{\partial u} = F \quad \Rightarrow \quad m\ddot{u} + ku = F(t)$




# Formulation of the equation of motion
## Method 2: The Lagrangian formalism

$$q(t)=\theta(t)$$
$$Q(t)=M(t)$$

$$\mathcal{K}=\frac{1}{2}ml^2\left(\dot{\theta}\right)^2,\ \ \mathcal{P}=mgl\left(1-\cos\left(\theta\right)\right)$$

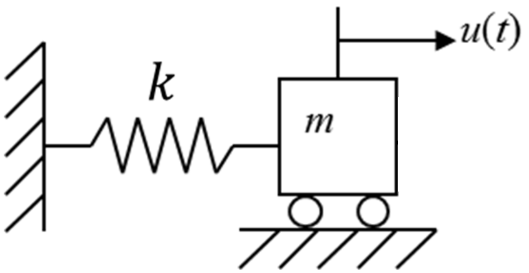
$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}}=\frac{d}{dt}\frac{\partial\mathcal{K}}{\partial\dot{\theta}}=\frac{d}{dt}\frac{\partial}{\partial\dot{\theta}}\left(\frac{1}{2}ml^2\left(\dot{\theta}\right)^2\right)=\frac{d}{dt}\left(ml^2\dot{\theta}\right)=ml^2\ddot{\theta}$$

$$\frac{\partial\mathcal{L}}{\partial\theta}=-\frac{\partial\mathcal{P}}{\partial\theta}=-\frac{\partial}{\partial\theta}\left(mgl\left(1-\cos\left(\theta\right)\right)\right)=-mgl\sin\left(\theta\right)\approx-mgl\theta$$

Equation of motion: $\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}}-\frac{\partial\mathcal{L}}{\partial\theta}=M \Rightarrow ml^2\ddot{\theta}+mgl\theta=M$

# Free vibration: problem statement

$$m\ddot{u} + ku = 0$$




Ordinary, homogeneous, linear differential equation of the second order with constant coefficients

$$u(0) = u_0, \ \dot{u}(0) = v_0$$

The initial conditions

# The general solution of the homogeneous equation of motion

$m\ddot{u} + ku = 0$ Homogeneous equation

$$u(t) = \sum_{n=1}^{2} U_n \exp(s_n t) = U_1 \exp(s_1 t) + U_2 \exp(s_2 t)$$ The general solution

$U_{1,2}$ and $s_{1,2}$ are complex-valued constants

# The characteristic equation and the eigenvalues

$$m\ddot{u} + ku = 0$$

$$m\frac{d^2}{dt^2}\left(\sum_{n=1}^{2} U_n \exp(s_n t)\right) + k\sum_{n=1}^{2} U_n \exp(s_n t) =$$

$$m\sum_{n=1}^{2} U_n s_n^2 \exp(s_n t) + k\sum_{n=1}^{2} U_n \exp(s_n t) =$$

$$\sum_{n=1}^{2} U_n \exp(s_n t)\left(m s_n^2 + k\right) = 0$$

$$\boxed{m s_n^2 + k = 0}$$

$$s_1 = \mathrm{i}\sqrt{k/m} = \mathrm{i}\omega_0, \qquad s_2 = -\mathrm{i}\sqrt{k/m} = -\mathrm{i}\omega_0$$

$$\omega_0 = \sqrt{k/m}$$

# The complex-valued and real-valued forms of the solution

$$u(t) = \sum_{n=1}^{2} U_n \exp(s_n t) = U_1 \exp(\mathrm{i}\omega_0 t) + U_2 \exp(-\mathrm{i}\omega_0 t)$$

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$$\exp(\pm \mathrm{i}\alpha) = \cos(\alpha) \pm \mathrm{i}\sin(\alpha)$$

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$$u(t) = U_1\left(\cos(\omega_0 t) + \mathrm{i}\sin(\omega_0 t)\right) + U_2\left(\cos(\omega_0 t) - \mathrm{i}\sin(\omega_0 t)\right) =$$

$$\cos(\omega_0 t)(U_1 + U_2) + \sin(\omega_0 t)(\mathrm{i}U_1 - \mathrm{i}U_2) =$$

$$A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

# The solution to the initial value problem

$$u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

$$u(0) = u_0,\ \dot{u}(0) = v_0$$

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$$u(0) = A\cos(0) + B\sin(0) = A$$

$$\dot{u}(0) = -\omega_0 A\sin(0) + B\omega_0\cos(0) = B\omega_0$$

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$$u(t) = u_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t)$$



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# The physical interpretation of the solution

$$u(t) = u_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t)$$

introduce: $u_0 = A_0 \cos(\varphi_0)$, $\frac{v_0}{\omega_0} = A_0 \sin(\varphi_0)$

$$u(t) = A_0 \cos(\omega_0 t - \varphi_0)$$

$$A_0 = \sqrt{u_0^2 + (v_0/\omega_0)^2}$$

$$\tan \varphi_0 = \left( \frac{v_0}{u_0 \omega_0} \right)$$

$A_0$ is the amplitude

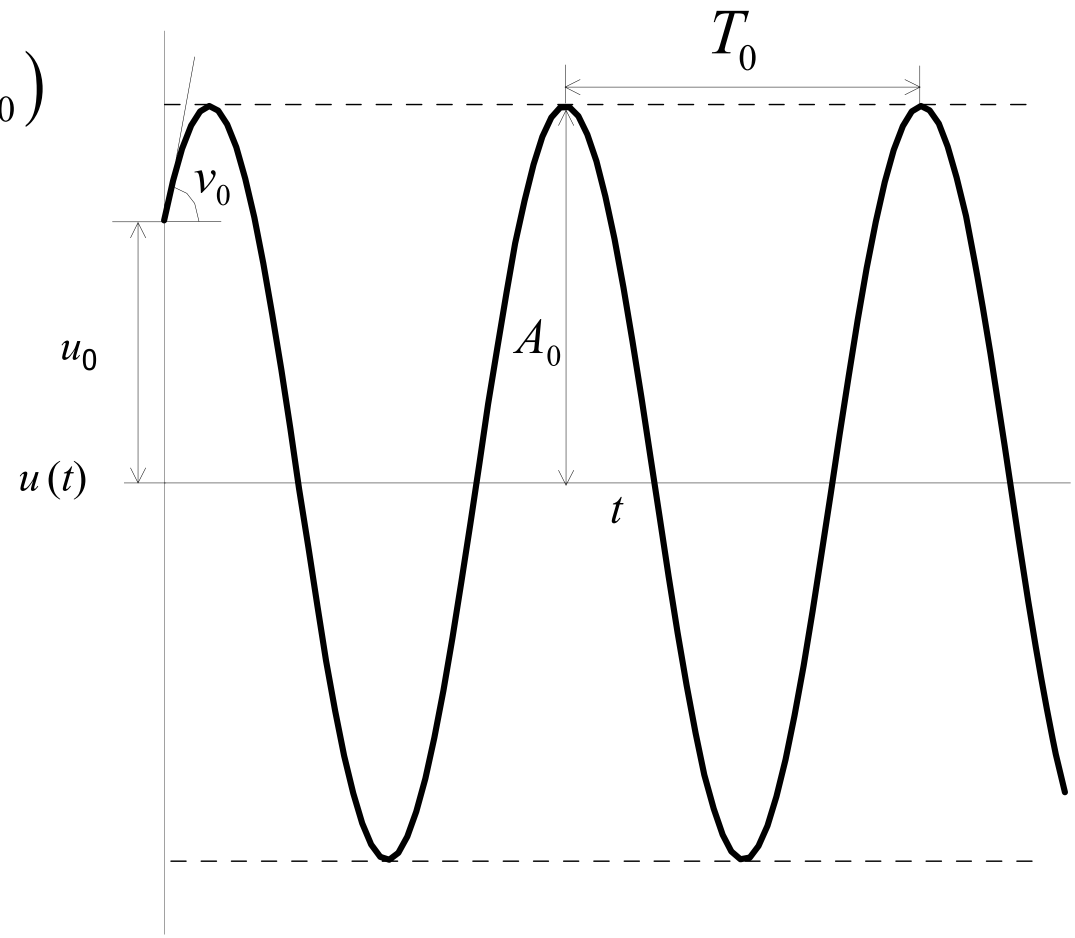
$\omega_0$ is the natural frequency

$T_0 = 2\pi/\omega_0$ is the natural period

$\varphi_0$ is the initial phase

# The physical interpretation of the solution

$$u(t) = A_0 \cos(\omega_0 t - \varphi_0)$$



# The energy of free vibration

$$\mathcal{E} = \mathcal{K} + \mathcal{P} = \frac{1}{2} m \left(\dot{u}\right)^2 + \frac{1}{2} k u^2$$

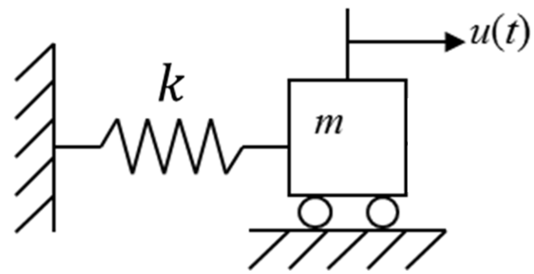
$$\mathcal{E} = \frac{1}{2} m \left( \left( -u_0 \omega_0 \sin\left(\omega_0 t\right) + v_0 \cos\left(\omega_0 t\right) \right)^2 + \omega_0^2 \left( u_0 \cos\left(\omega_0 t\right) + \frac{v_0}{\omega_0} \sin\left(\omega_0 t\right) \right)^2 \right)$$

$$\boxed{\mathcal{E} = \frac{1}{2} m \left( u_0^2 \omega_0^2 + v_0^2 \right) = \frac{1}{2} m v_0^2 + \frac{1}{2} k u_0^2}$$

The energy is conserved: it remains constant and equal to the initial energy of the system

# Exercise: determine free vibration

The free vibration of a single-degree-of-freedom system is governed by the following differential equation:




$$m\ddot{u} + ku = 0$$

Q1: Derive the general solution to this equation of motion.
Now, assume that the initial conditions are

$$u(0) = 0, \;\; \dot{u}(0) = v_0$$

Q2: Determine the free vibration of the system for the given initial conditions and plot the result versus time.