Dynamics of Systems

CTB 2300

Dr. Karel N. van Dalen
Dr. Hayo Hendrikse
Prof. Andrei V. Metrikine

Department: Engineering Structures
Research group: Dynamics of Solids & Structures
Room: 3.61
E-mail: K.N.vanDalen@tudelft.nl
Web: www.tudelft.nl/knvandalen

Department: Hydraulic Engineering
Research group: Offshore Engineering
Room: 3.71
E-mail: H.Hendrikse@tudelft.nl
Web: tinyurl.com/hhendrikse

Lecture 6

CTB 2300 Dynamics of Systems
Faculty of Civil Engineering and Geosciences
2022 Delft, The Netherlands

Delft University of Technology

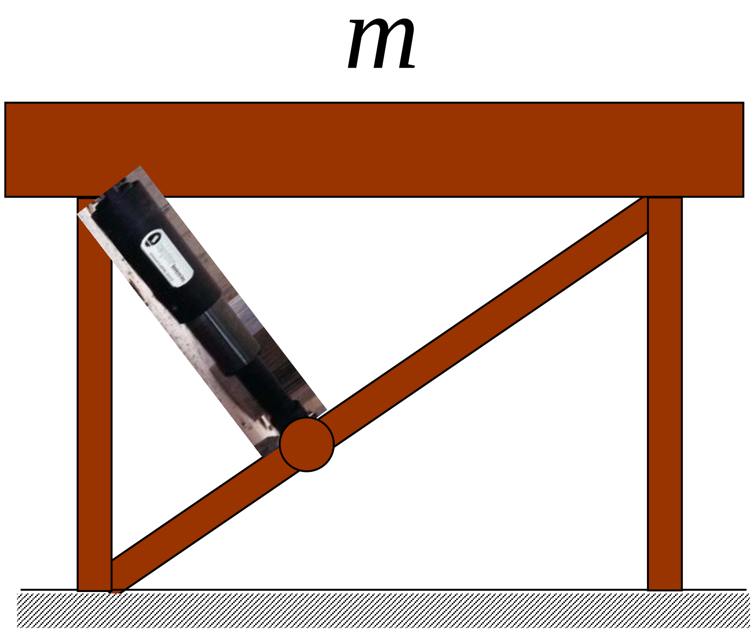
Contents of Lecture 6

1. **Free vibration of an SDOF with viscous damping:**
    - Viscous dampers: impression
    - Derivation of the equation of motion
    - The characteristic equation and motion types
    - Super-critically damped motion
    - Critically damped motion
    - Sub-critically damped vibration
    - Measurement of the damping ratio

Viscous dampers: impression

Viscous damper in a building

Viscous dampers: impression



Viscous damper in a frame structure

Video: Dampers for earthquake protection

Viscous dampers: impression

Seismic fluid viscous dampers for large highway bridge, 1.5 million pounds output force (how large they can be!)

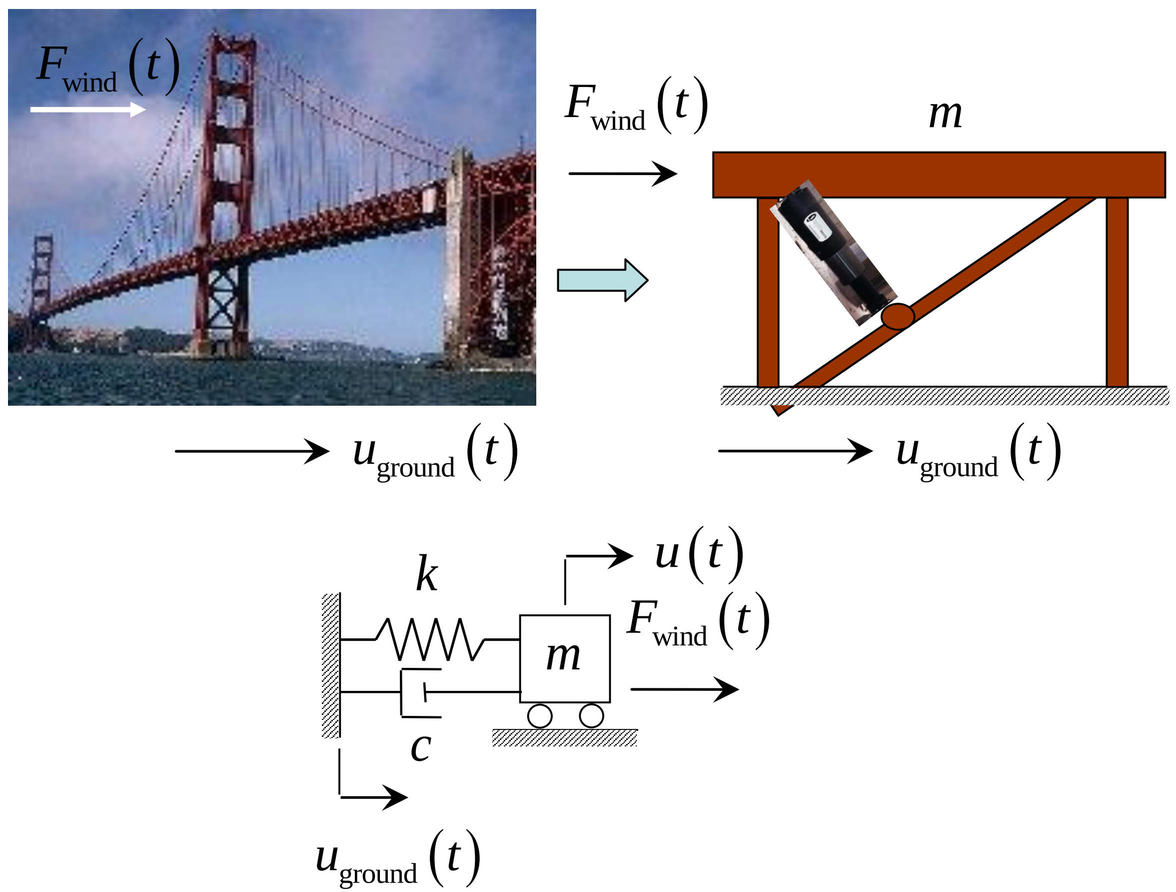
Viscous dampers: impression

Viscous dampers at a cable anchorage for a cable-stayed bridge

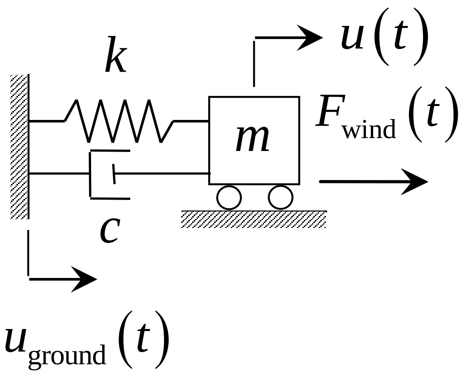
Viscous dampers: impression

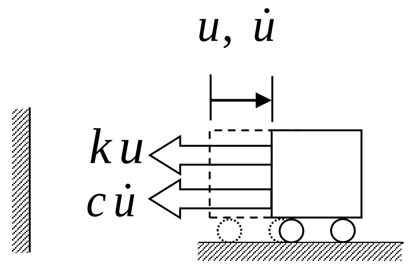
Viscous dampers for the Millennium bridge in London

SDOF with viscous damping

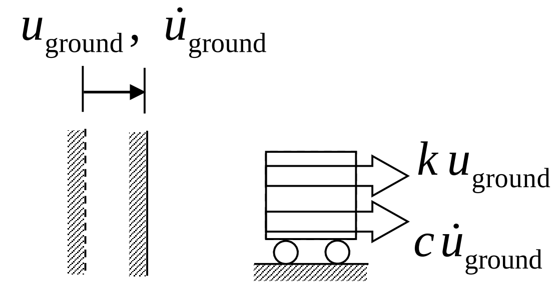


Derivation of the equation of motion





First admissible motion and corresponding forces



Second admissible motion and corresponding forces

Resulting equation of motion:

$$m\ddot{u} = -k\,u + k\,u_{\text{ground}} - c\,\dot{u} + c\,\dot{u}_{\text{ground}} + F_{wind}(t)$$

$$\Rightarrow$$

$$m\,\ddot{u} + c\,\dot{u} + k\,u = F(t),$$

$$F(t) = k\,u_{\text{ground}} + c\,\dot{u}_{\text{ground}} + F_{wind}(t)$$

$$\text{mass} \times \text{acceleration} = \text{force}$$

Damped SDOF: Generalized equation of motion and the set of governing equations

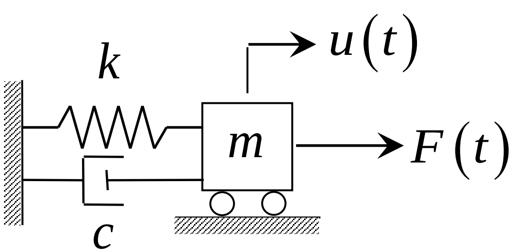
The governing equations for damped SDOF systems:

$$m\,\ddot{u} + c\,\dot{u} + k\,u = F(t)$$

⟸ **Equation of motion**

$$\left.\begin{aligned} u(0) &= u_0 \\ \dot{u}(0) &= v_0 \end{aligned}\right\}$$

⟸ **Initial conditions**



⟸ **Schematization of a generalized SDOF with viscous damping (the damped SDOF)**

CTB 2300 Dynamics of Systems
Faculty of Civil Engineering and Geosciences
2022 Delft, The Netherlands

Free vibration of the damped SDOF: equation of motion

Equation of motion:

$$m\,\ddot{u} + c\,\dot{u} + k\,u = 0$$

Canonical form of equation of motion:

$$\boxed{\ddot{u} + 2\zeta\,\omega_0 \dot{u} + \omega_0^2\,u = 0,}$$

$\omega_0 = \sqrt{k/m}$ is the natural frequency

$\zeta = c/c_{\text{critical}}$ is the damping ratio

$c_{\text{critical}} = 2\sqrt{k\,m}$ is the critical damping

The characteristic equation and possible motion types

Equation of motion:

$$\ddot{u} + 2\zeta\,\omega_0 \dot{u} + \omega_0^2\, u = 0,$$

The general solution:

$$u(t) = \sum_{n=1}^{2} U_n \exp(s_n t)$$

Characteristic equation and its roots:

$$s_n^2 + 2\zeta\,\omega_0 s_n + \omega_0^2 = 0, \qquad s_{1,2} = \omega_0\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

Possible types of motion:

aperiodic motion: $\begin{cases} \zeta > 1 \text{ (super-critically damped system, I)} \\ \zeta = 1 \text{ (critically damped system, II)} \end{cases}$

damped vibration: $\zeta < 1$ (sub-critically damped system, III)

I. The super-critically damped system (aperiodic motion)

The roots of the characteristic equation:

$$s_{1,2} = \omega_0\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right) \text{ are real and negative}$$

The general solution to the equation of motion:

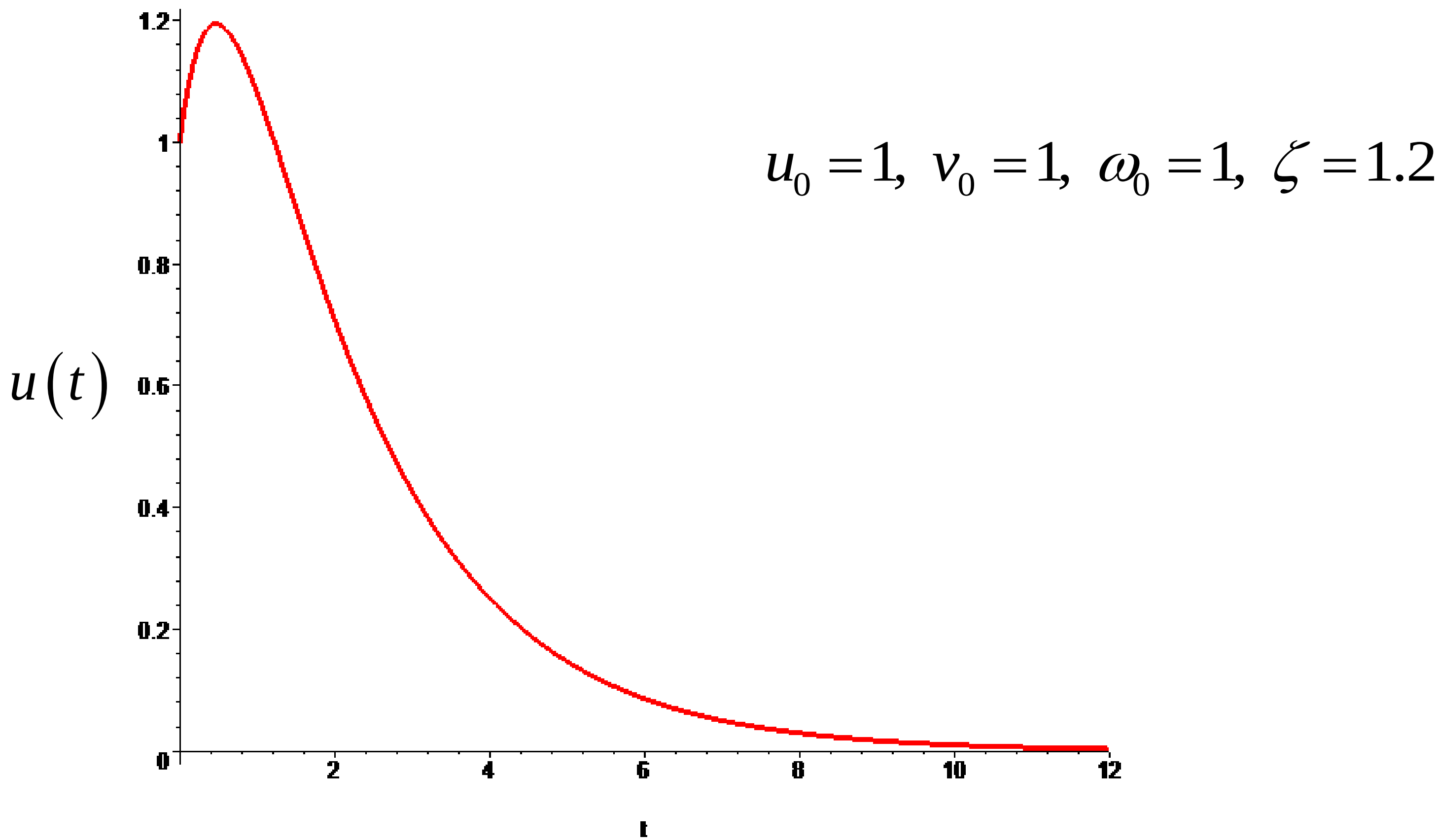
$$u(t) = \exp(-\zeta\omega_0 t)\left(A\exp\left(\omega_0 t\sqrt{\zeta^2 - 1}\right) + B\exp\left(-\omega_0 t\sqrt{\zeta^2 - 1}\right)\right)$$

The solution to the governing equations (with the initial conditions):

$$u(t) = \frac{\exp(-\zeta\omega_0 t)}{2\sqrt{\zeta^2 - 1}}\left\{\left(u_0\omega_0\left(\sqrt{\zeta^2 - 1} + \zeta\right) + v_0\right)\exp\left(\omega_0 t\sqrt{\zeta^2 - 1}\right)\right.$$

$$\left. + \left(u_0\omega_0\left(\sqrt{\zeta^2 - 1} - \zeta\right) - v_0\right)\exp\left(-\omega_0 t\sqrt{\zeta^2 - 1}\right)\right\}$$

I. The super-critically damped system (aperiodic motion)



To understand the effect of the parameters on the motion, please use the MAPLE file to this lecture

II. The critically damped system (aperiodic motion)

The roots of the characteristic equation:

$s_{1,2} = -\omega_0$ are the same (multiple); real and negative

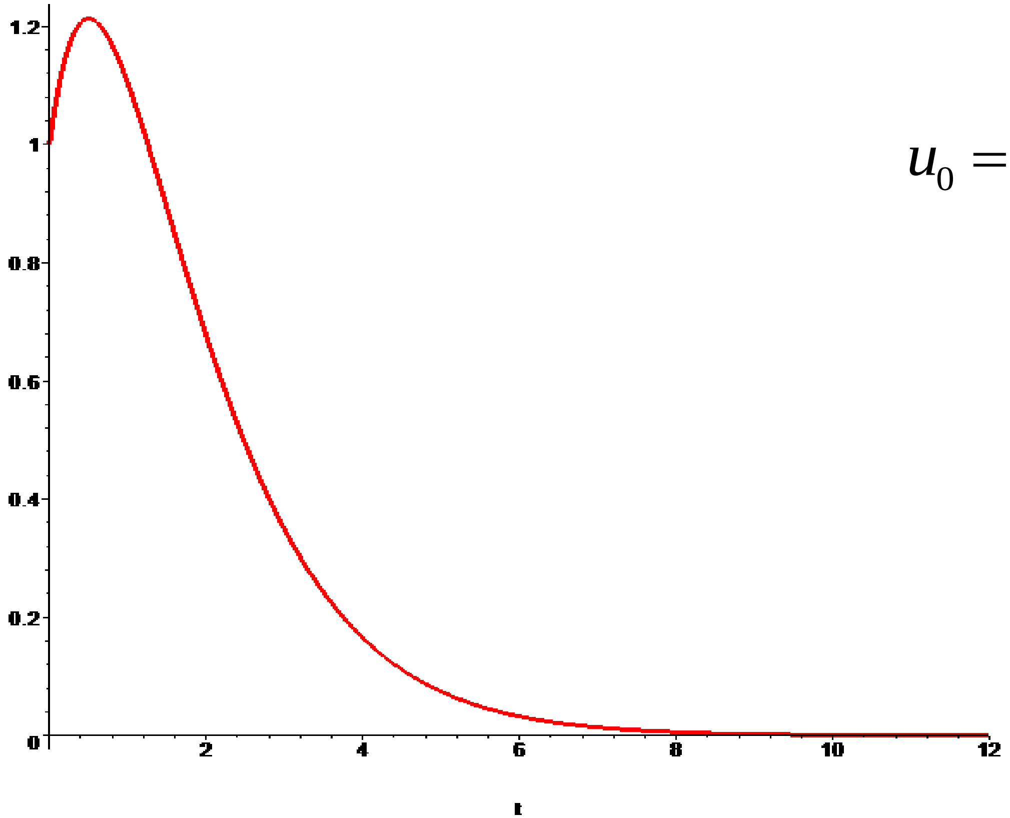
The general solution to the equation of motion:

$$u(t) = \exp(-\omega_0 t)(A + Bt)$$

The solution to the governing equations (with the initial conditions):

$$u(t) = \exp(-\omega_0 t)\left\{u_0 + (v_0 + \omega_0 u_0)t\right\}$$

II. The critically damped system (aperiodic motion)



$$u_0 = 1,\ v_0 = 1,\ \omega_0 = 1$$

To understand the effect of the parameters on the motion, please use the MAPLE file to this lecture

III. The sub-critically damped system (damped vibration)

The roots of the characteristic equation:

$s_{1,2} = \omega_0\left(-\zeta \pm \mathrm{i}\sqrt{1-\zeta^2}\right)$ are complex-conjugate (real part is negative)

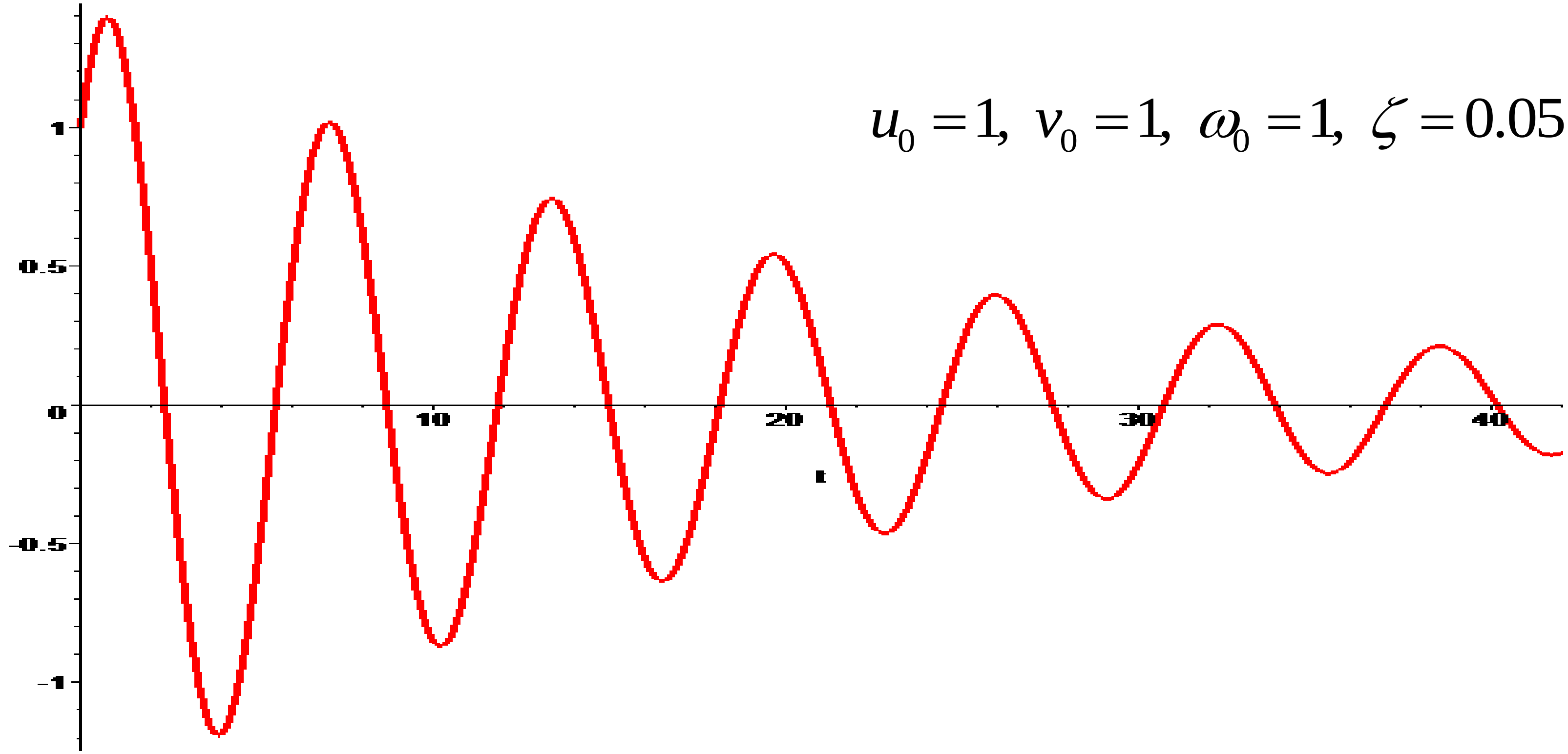
The general solution to the equation of motion:

$$u(t) = \exp(-\zeta\omega_0 t)\left(A\cos(\omega_1 t) + B\sin(\omega_1 t)\right), \qquad \omega_1 = \omega_0\sqrt{1-\zeta^2}$$

The solution to the governing equations (with the initial conditions):

$$u(t) = \exp(-\zeta\omega_0 t)\left(u_0\cos(\omega_1 t) + \frac{v_0 + \zeta\omega_0 u_0}{\omega_1}\sin(\omega_1 t)\right)$$

III. The sub-critically damped system (damped vibration)



To understand the effect of the parameters on the motion, please use the MAPLE file to this lecture

CTB 2300 Dynamics of Systems
Faculty of Civil Engineering and Geosciences
2022 Delft, The Netherlands

Delft University of Technology

III. Damped vibration: a compact form of the solution

The solution:

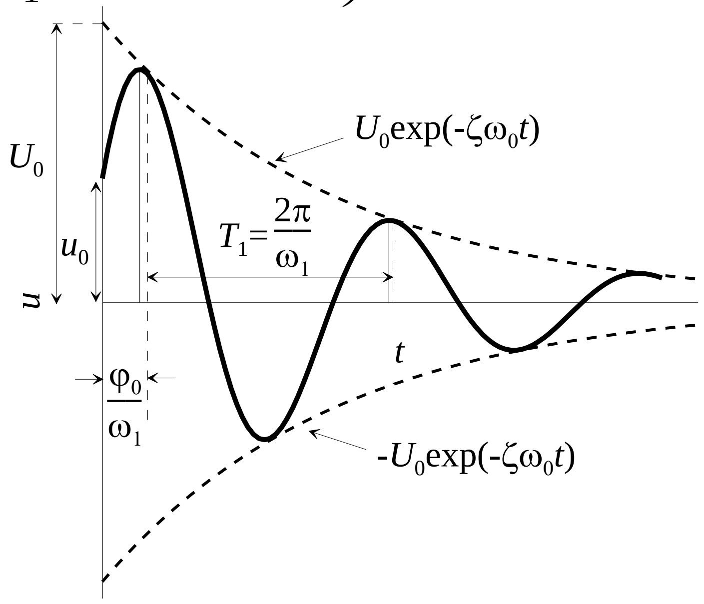
$$u(t) = \exp(-\zeta\omega_0 t)\left(u_0\cos(\omega_1 t) + \frac{v_0 + \zeta\omega_0 u_0}{\omega_1}\sin(\omega_1 t)\right)$$

Alternative form of the solution:

$$u(t) = U_0 \exp(-\zeta\omega_0 t)\cos(\omega_1 t + \varphi_0)$$

$$U_0 = \sqrt{u_0^2 + \left(\frac{v_0 + \zeta\omega_0 u_0}{\omega_1}\right)^2}$$

$$\varphi_0 = -\arctan\left(\frac{v_0 + \zeta\omega_0 u_0}{u_0\omega_1}\right)$$



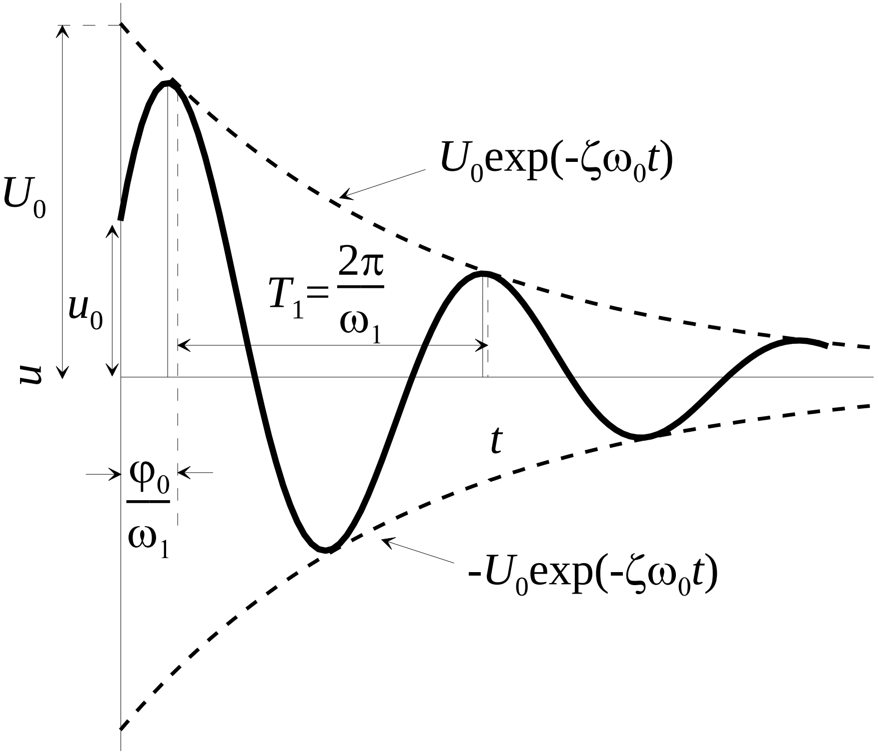
III. Damped vibration: the amplitude, damped frequency, period and phase angle

$$u(t) = U_0 \exp(-\zeta\omega_0 t)\cos(\omega_1 t + \varphi_0)$$

Amplitude: $U_0 \exp(-\zeta\omega_0 t)$

Period: $T_1 = 2\pi/\omega_1 = \dfrac{2\pi}{\omega_0\sqrt{1-\zeta^2}}$

Phase angle: φ_0



III. Damped vibration: how can we measure the damping ratio?

Measurement of damping.

Let u_1 be a displacement at t

and u_s a displacement after $(s-1)$ cycles.

Then

$$\frac{u_1}{u_s} = \exp\left(\left(s-1\right)\zeta\omega_0 T_1\right)$$

or $$\log\left(\frac{u_1}{u_s}\right) = \left(s-1\right)\zeta\omega_0 T_1$$

$$\zeta\omega_0 T_1 = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} - \text{logarithmic decrement}$$

$$u\left(t\right) = U_0 \exp\left(-\zeta\omega_0 t\right)\cos\left(\omega_1 t + \varphi_0\right)$$

The frequency of damped vibrations is normally approximated by the natural frequency of the undamped system

The reason:

$$\omega_1 = \omega_0\sqrt{1-\zeta^2} \approx \omega_0\left(1-\frac{1}{2}\zeta^2\right) \approx \omega_0$$

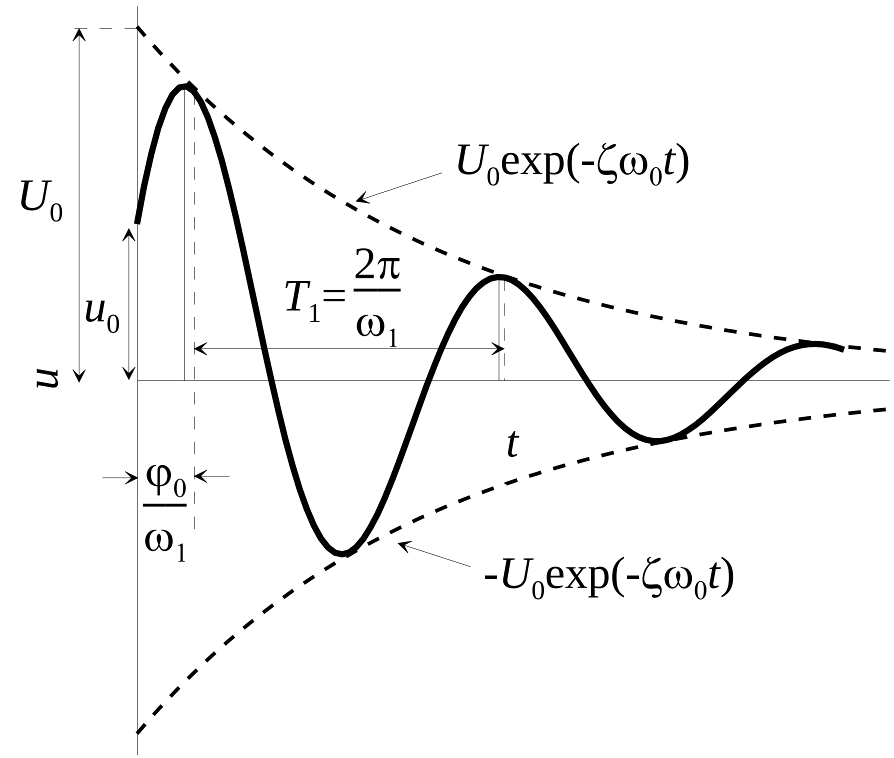
Exercise: compute the damping ratio from observation

We consider the free vibration of a sub-critically damped SDOF system, as shown in the figure.

Suppose that the amplitude decay over 1 natural period is equal to α.

Q1: derive the damping ratio ζ in terms of α.

Q2: is the result influenced by the specific initial conditions that excited the motion?



$$u(t) = U_0 \exp(-\zeta\omega_0 t)\cos(\omega_1 t + \varphi_0)$$

CTB 2300 Dynamics of Systems
Faculty of Civil Engineering and Geosciences
2022 Delft, The Netherlands

Delft University of Technology