Dynamics of Systems

CTB 2300

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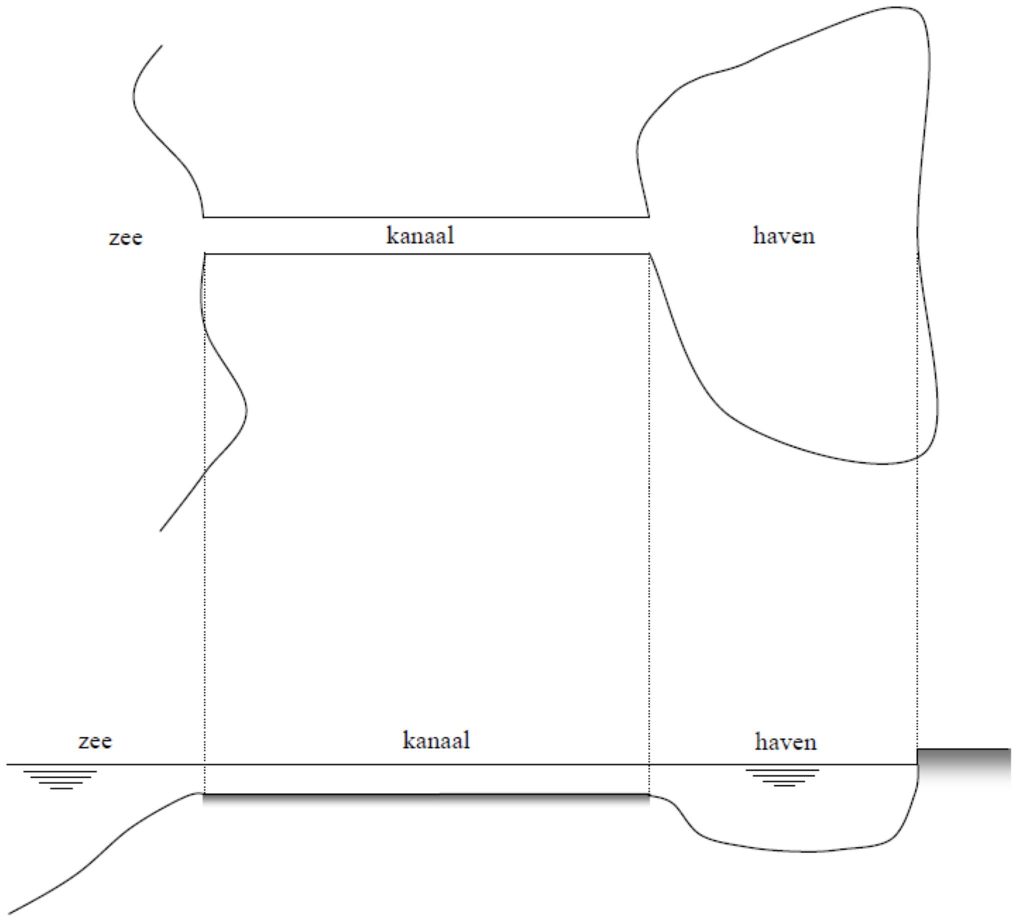
Lecture 12

Contents of Lecture 12

1. Equation of motion for the water level in a harbour connected with Sea by a canal
2. Analogy with the mechanical oscillations
3. Analysis of the hydraulic systems based on the solutions obtained for the mechanical systems
4. Other examples of oscillatory dynamical systems

Dynamics of the water level in a harbour



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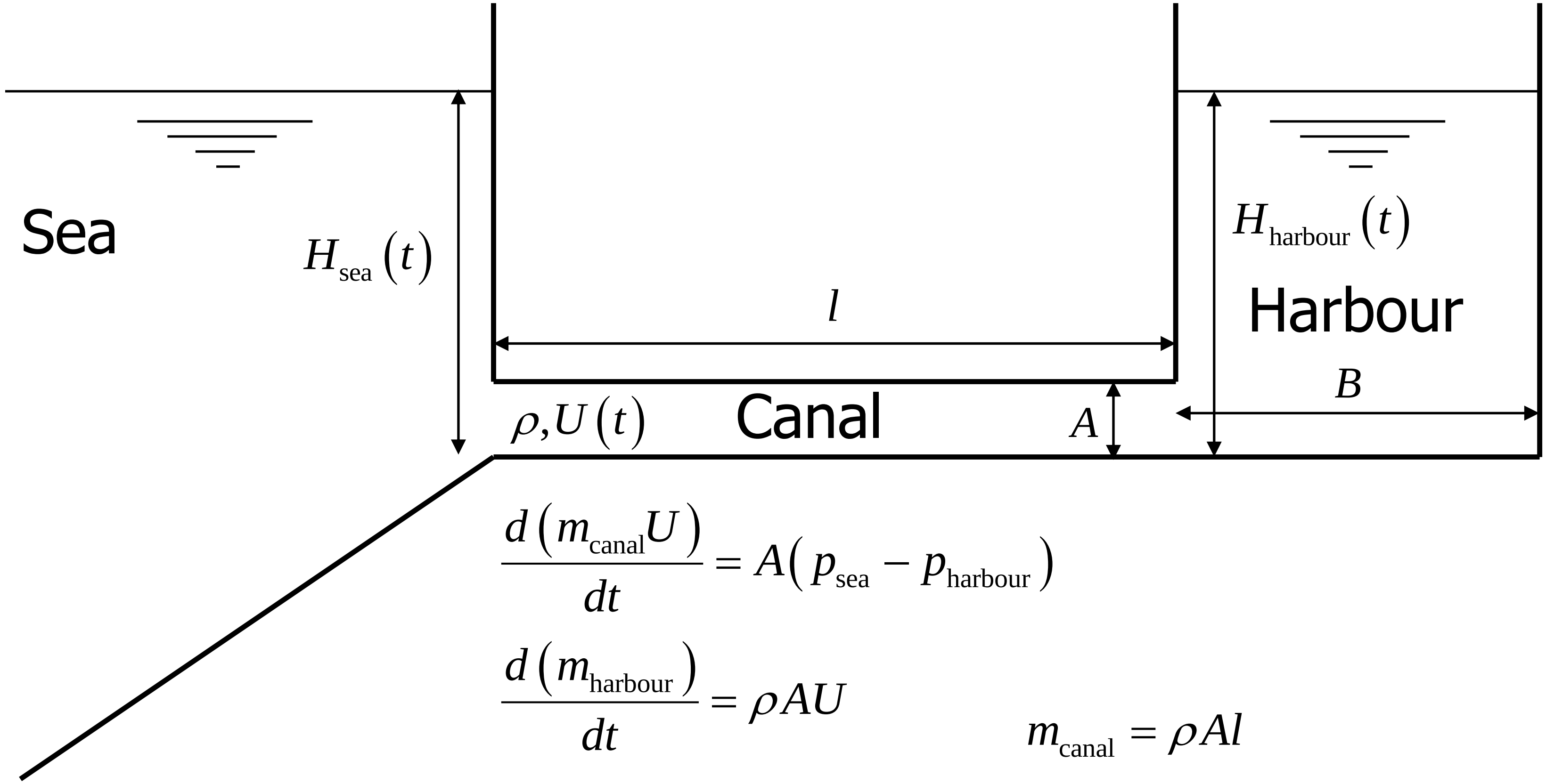
Dynamics of the water level in a harbour



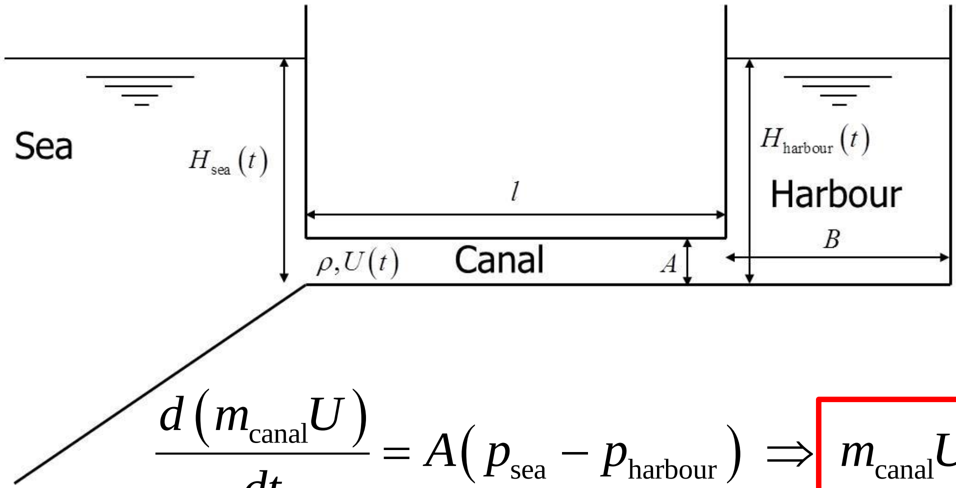
$$\frac{d\left(m_{canal}U\right)}{dt} = A\left(p_{sea} - p_{harbour}\right)$$

$$\frac{d\left(m_{harbour}\right)}{dt} = \rho A U$$

$$m_{canal} = \rho A l$$

$$m_{harbour} = \rho B H_{harbour}$$

Dynamics of the water level in a harbour



$$\frac{d(m_{canal}U)}{dt} = A(p_{sea} - p_{harbour}) \Rightarrow \boxed{m_{canal}\dot{U} = A(\rho g H_{sea} - \rho g H_{harbour})}$$

$$\begin{cases} \dfrac{d(m_{harbour})}{dt} = \rho A U \\ m_{harbour} = \rho B H_{harbour} \end{cases} \Rightarrow \rho B \dot{H}_{harbour} = \rho A U \Rightarrow B\dot{H}_{harbour} = AU \Rightarrow \boxed{U = \frac{B}{A}\dot{H}_{harbour}}$$

$$\begin{cases} m_{canal}\dot{U} = A(\rho g H_{sea} - \rho g H_{harbour}) \\ m_{canal} = \rho A l \end{cases} \Rightarrow \boxed{(\rho A l)\frac{B}{A}\ddot{H}_{harbour} = A(\rho g H_{sea} - \rho g H_{harbour})}$$

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Dynamics of the water level in a harbour

$$(\rho A l)\frac{B}{A}\ddot{H}_{\text{harbour}} = A(\rho g H_{\text{sea}} - \rho g H_{\text{harbour}}) \Rightarrow \rho B l \ddot{H}_{\text{harbour}} + \rho A g H_{\text{harbour}} = \rho A g H_{\text{sea}}$$

$$\rho B \ddot{H}_{\text{harbour}} + \rho A \frac{g}{l} H_{\text{harbour}} = \rho A \frac{g}{l} H_{\text{sea}}$$

ρB - mass of water in the harbour per unit height

$\rho A \frac{g}{l}$ - force from the harbour on the water in the canal per unit height (equivalent spring constant)

$\rho A \frac{g}{l}$ - force from the sea on the water in the canal per unit height

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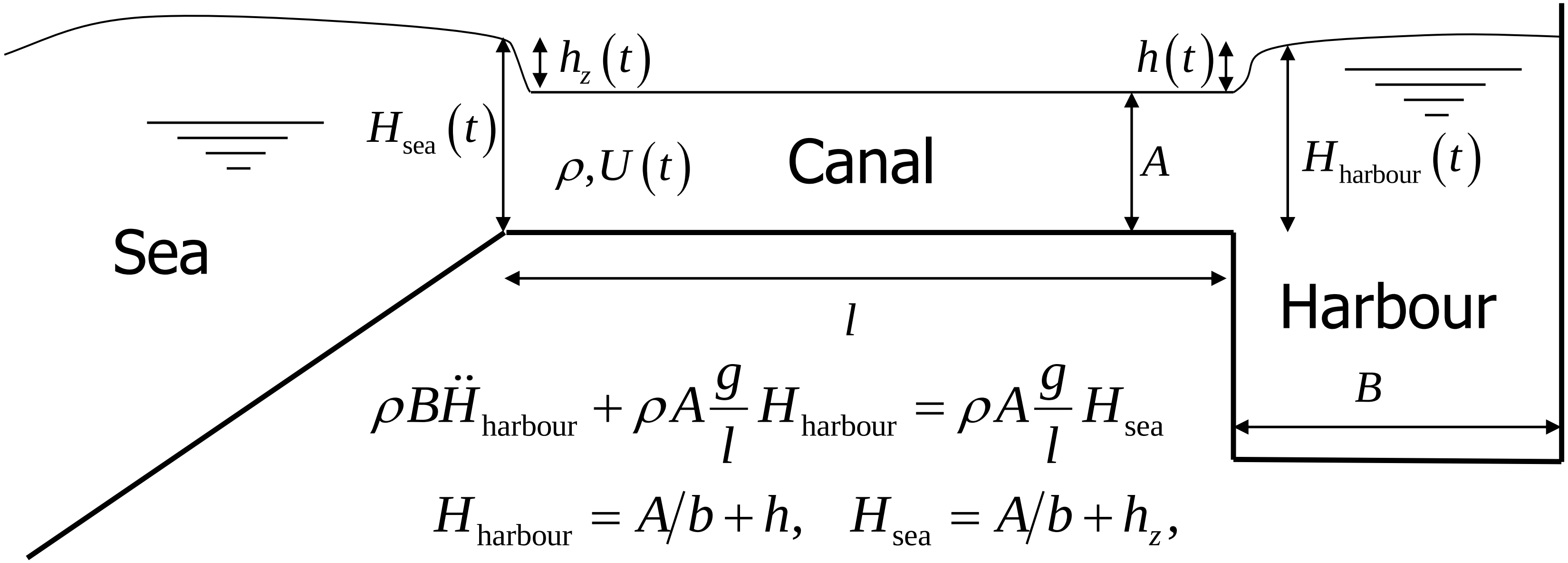
Dynamics of the water level in a harbour



$$\rho B\ddot{H}_{harbour} + \rho A\frac{g}{l}H_{harbour} = \rho A\frac{g}{l}H_{sea}$$

$$H_{harbour} = A/b + h, \quad H_{sea} = A/b + h_z,$$

b - width of the canal

$$\rho B\ddot{h} + \rho A\frac{g}{l}\left(\frac{A}{b} + h\right) = \rho A\frac{g}{l}\left(\frac{A}{b} + h_z\right)$$

$$\rho B\ddot{h} + \rho A\frac{g}{l}h = \rho A\frac{g}{l}h_z$$

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Analogy with the mass-spring system subject to a force

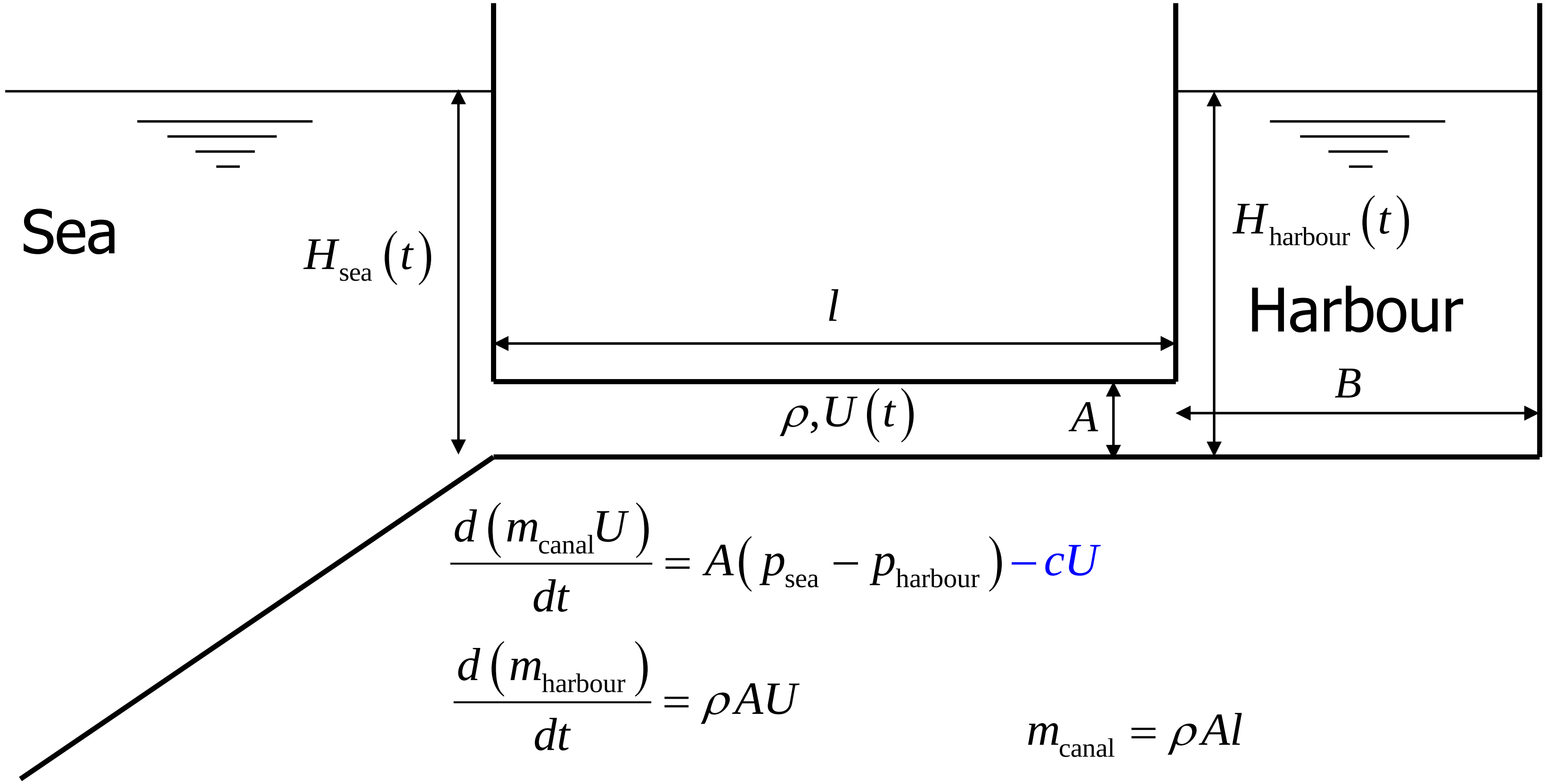
$$\rho B\ddot{h} + \rho A\frac{g}{l}h = \rho A\frac{g}{l}h_z$$ Equations of motion $$m\ddot{u} + ku = F_u$$

$$\rho B \leftrightarrow m \qquad (\text{mass per unit height} \leftrightarrow \text{mass})$$

$$\rho A\frac{g}{l} \leftrightarrow k \qquad (\text{equivalent stiffness} \leftrightarrow \text{stiffness})$$

$$\rho A\frac{g}{l}h_z \leftrightarrow F_u \qquad (\text{force} \leftrightarrow \text{force})$$

Effect of the friction with the canal bottom



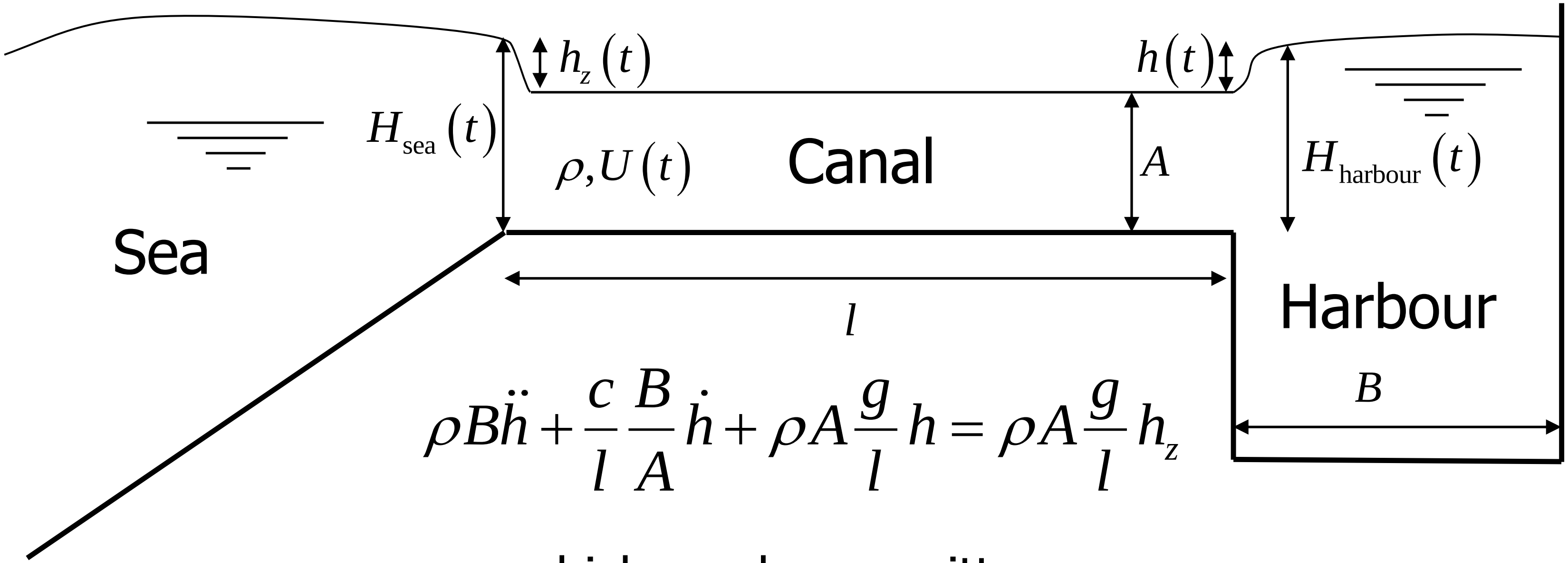
$$\frac{d\left(m_{\text{canal}}U\right)}{dt} = A\left(p_{\text{sea}} - p_{\text{harbour}}\right) - cU$$

$$\frac{d\left(m_{\text{harbour}}\right)}{dt} = \rho AU$$

$$m_{\text{canal}} = \rho Al$$

$$m_{\text{harbour}} = \rho BH_{\text{harbour}}$$

Effect of the friction with the canal bottom



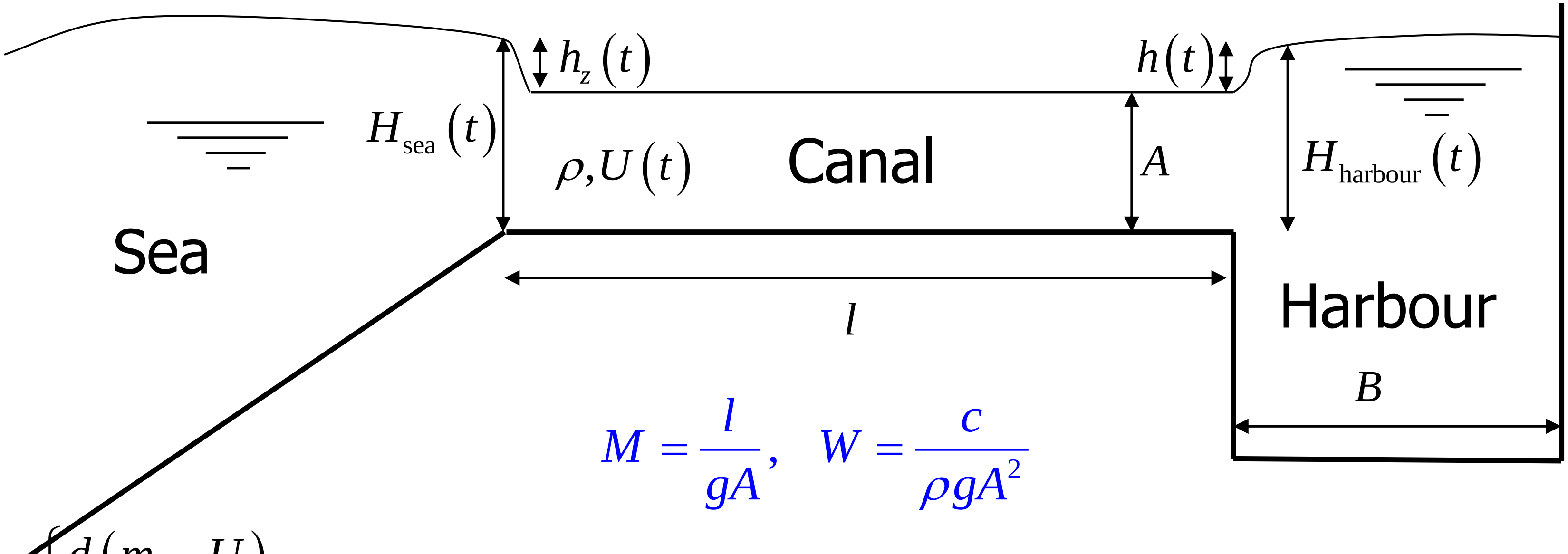
$$\rho B\ddot{h} + \frac{c}{l}\frac{B}{A}\dot{h} + \rho A\frac{g}{l}h = \rho A\frac{g}{l}h_z$$

which can be re-written as

$$BM\ddot{h} + BW\dot{h} + h = h_z,$$

$$M = \frac{l}{gA}, \quad W = \frac{c}{\rho g A^2}$$

Some specific hydraulic definitions



$$M = \frac{l}{gA}, \quad W = \frac{c}{\rho g A^2}$$

$$\begin{cases} \dfrac{d\left(m_{canal} U\right)}{dt} = A\left(p_{sea} - p_{harbour}\right) - cU \\ m_{canal} = \rho A l \end{cases} \Rightarrow \quad M\dot{Q} + WQ = h_z - h \qquad \left(Q = AU\right)$$

$$\begin{cases} \dfrac{d\left(m_{harbour}\right)}{dt} = \rho A U \\ m_{harbour} = \rho B H_{harbour} \end{cases} \Rightarrow \quad B\dot{h} = Q$$

$$BM\ddot{h} + BW\dot{h} + h = h_z$$

Discharge (debiet): Q

Inertia (traagheid): M

How to analyze?

$$BM\ddot{h} + BW\dot{h} + h = h_z$$

$$\Downarrow$$

$$\ddot{h} + \frac{W}{M}\dot{h} + \frac{1}{BM}h = \frac{1}{BM}h_z$$

$$\Downarrow$$

$$\ddot{h} + 2\zeta\omega_0\dot{h} + \omega_0^2 h = f(t)$$

$$\omega_0^2 = \frac{1}{BM}, \quad \zeta = \frac{W}{2}\sqrt{\frac{B}{M}}, \quad f(t) = \frac{h_z(t)}{BM}$$

We can simply use all formulas obtained for the mass-spring-dashpot system

$$\ddot{h} + 2\zeta\omega_0\dot{h} + \omega_0^2 h = f(t)$$

$$\omega_0^2 = \frac{1}{BM}$$

$$\zeta = \frac{W}{2}\sqrt{\frac{B}{M}}$$

$$f(t) = \frac{h_z(t)}{BM}$$

hydraulic system

$$\ddot{u} + 2\zeta\omega_0\dot{u} + \omega_0^2 u = f(t)$$

$$\omega_0^2 = \frac{k}{m}$$

$$\zeta = \frac{c}{2\sqrt{km}}$$

$$f(t) = \frac{F(t)}{m}$$

mechanical system

Example 1: free vibration of the hydraulic system

As derived in Lecture 6, slide 19, the sub-critically damped free vibration of a mechanical system is described by

$$u(t) = \exp(-\zeta\omega_0 t)\left(u_0 \cos(\omega_1 t) + \frac{v_0 + \zeta\omega_0 u_0}{\omega_1}\sin(\omega_1 t)\right)$$

Therefore, using the expressions given on the previous slide, we find the following expression for the hydraulic system

$$h(t) = \exp(-\zeta\omega_0 t)\left(h(0)\cos(\omega_1 t) + \frac{\dot{h}(0) + \zeta\omega_0 h(0)}{\omega_1}\sin(\omega_1 t)\right)$$

$$\omega_0^2 = \frac{1}{BM},\ \zeta = \frac{W}{2}\sqrt{\frac{B}{M}},\ \omega_1 = \omega_0\sqrt{1-\zeta^2} = \frac{1}{\sqrt{BM}}\sqrt{1-\frac{W^2}{4}\frac{B}{M}}$$

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Example 2: amplitude of the steady-state vibration of the hydraulic system caused by $h_z(t) = h_{z0}\cos(\omega t)$

As derived in Lecture 7, slide 11, the amplitude of the steady-state response of the corresponding mechanical system is given as

$$U_{\text{steady}} = \frac{u_{\text{static}}}{\sqrt{\left(1-\omega^2/\omega_0^2\right)^2 + 4\zeta^2\,\omega^2/\omega_0^2}}, \qquad u_{\text{static}} = \frac{F_0}{k} = \frac{f_0}{\omega_0^2}$$

Therefore, using the relevant expressions for the hydraulic system (slide 13 of the current lecture), we obtain

$$H_{\text{steady}} = \frac{f_0}{\omega_0^2}\frac{1}{\sqrt{\left(1-\omega^2/\omega_0^2\right)^2 + 4\zeta^2\,\omega^2/\omega_0^2}}$$

$$\omega_0^2 = \frac{1}{BM}, \;\; \zeta = \frac{W}{2}\sqrt{\frac{B}{M}}, \;\; f_0 = \frac{h_{z0}}{BM}$$

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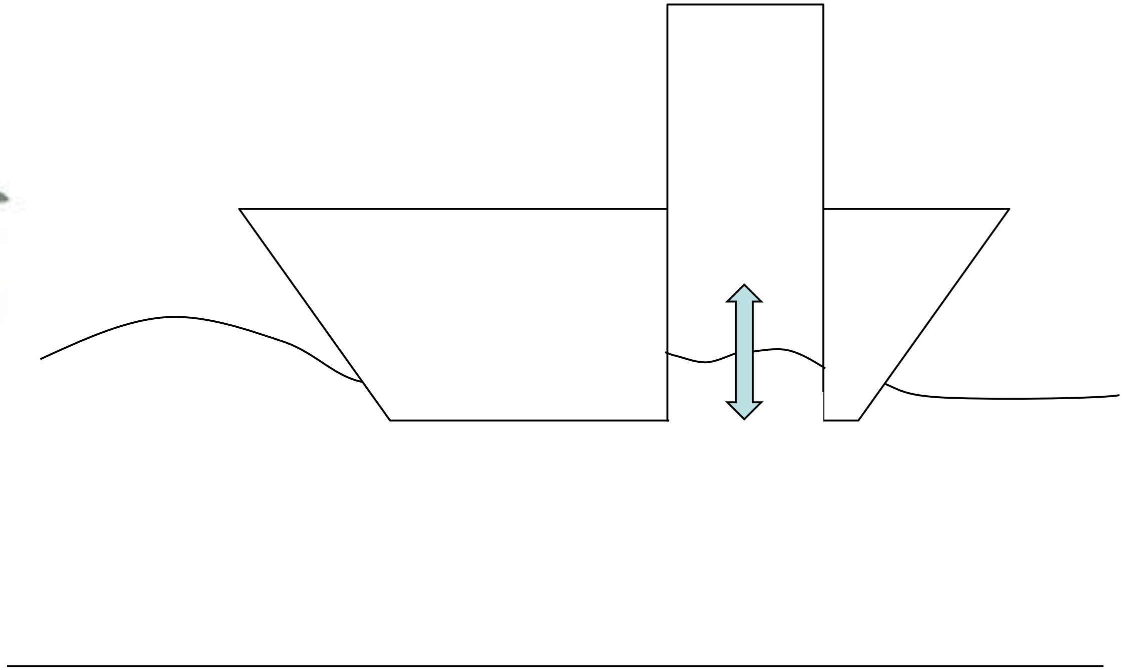
Other examples of SDOF dynamical systems: pressure fluctuations in an enclosed moonpool of an arctic drillship



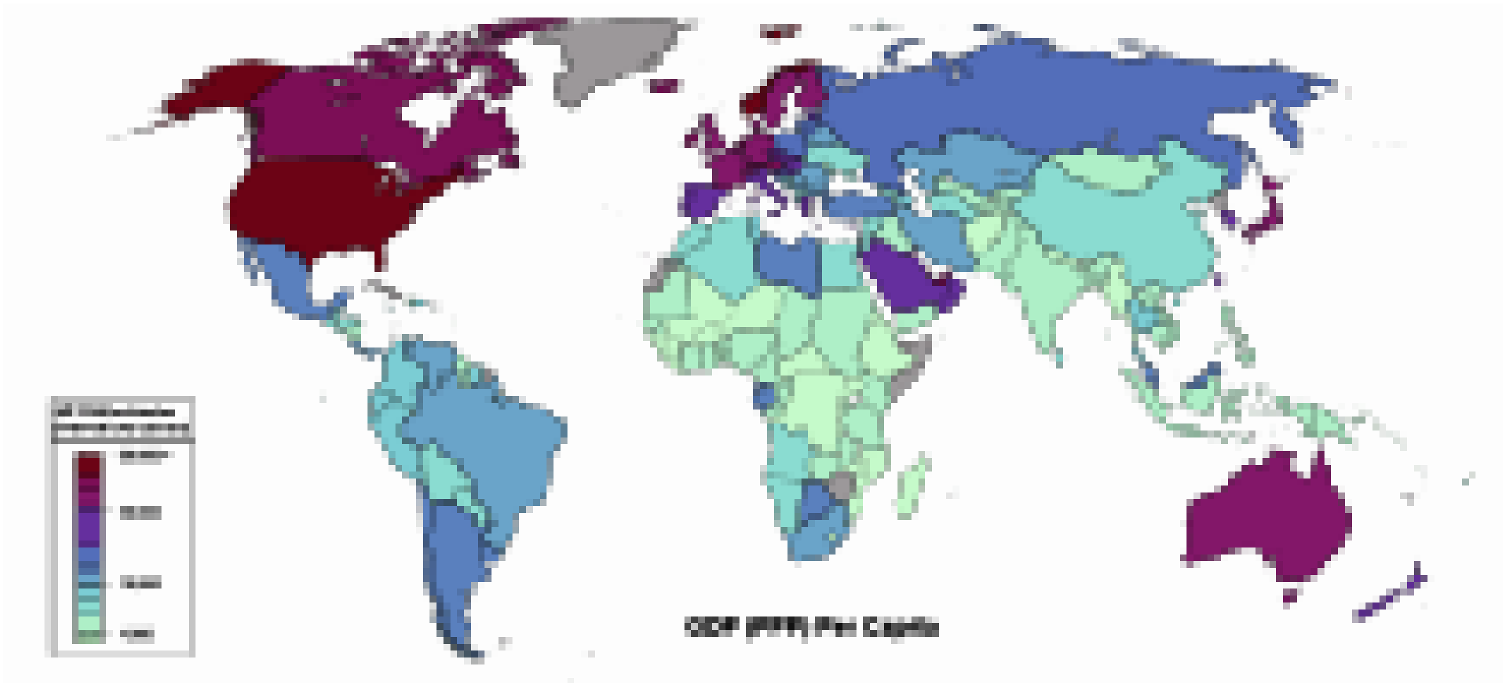
Other examples of SDOF dynamical systems: the Helmholtz resonator (sound trapper)

$$\omega_0 = c_{\text{sound}} \sqrt{\frac{A}{V_0 L}}$$

Other examples of SDOF dynamical systems: a loudspeaker

Other examples of SDOF dynamical systems: voice production

Other examples of SDOF dynamical systems: business cycle



Other examples of SDOF dynamical systems: predator-pray interaction

$\dot{x} = bx - pxy \;\; (\text{pray})$

$x(t) - \text{Number of the prays (rabbits)}$

$\dot{y} = -dy + rxy \;\; (\text{predator})$

$y(t) - \text{Number of the predators (foxes)}$