Dynamics of Systems

CTB 2300

<u>Dr. Karel N. van Dalen</u>

Dr. Hayo Hendrikse

Prof. Andrei V. Metrikine

Department: Engineering Structures
Research group: Dynamics of Solids & Structures
Room: 3.61
E-mail: K.N.vanDalen@tudelft.nl
Web: www.tudelft.nl/knvandalen

Department: Hydraulic Engineering
Research group: Offshore Engineering
Room: 3.71
E-mail: H.Hendrikse@tudelft.nl
Web: tinyurl.com/hhendrikse

Lecture 7

Contents of Lecture 7

Forced vibration of an SDOF with viscous damping subject to a harmonic force:

- The general solution in the real and complex forms
- The steady-state response
- The magnification (dynamic amplification) factor
- The phase lag

Video: Dampers for earthquake protection

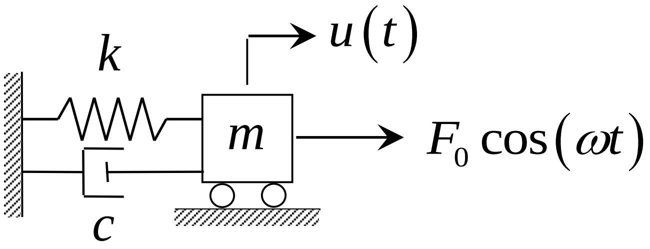
CTB 2300 Dynamics of Systems
Faculty of Civil Engineering and Geosciences
2022 Delft, The Netherlands

Delft University of Technology

An SDOF with viscous damping under a harmonic force: statement of the problem

$$m\,\ddot{u} + c\,\dot{u} + k\,u = F_0 \cos(\omega t)$$ ⟸ **Equation of motion**

$$\left.\begin{aligned} u(0) &= u_0 \\ \dot{u}(0) &= v_0 \end{aligned}\right\}$$ ⟸ **Initial conditions**



CTB 2300 Dynamics of Systems
Faculty of Civil Engineering and Geosciences
2022 Delft, The Netherlands

Delft University of Technology

An SDOF with viscous damping under a harmonic force: canonical form of the equation of motion

Canonical form of the equation of motion:

$$\ddot{u} + 2\zeta\,\omega_0\dot{u} + \omega_0^2\,u = f_0\cos(\omega t),$$

$\omega_0 = \sqrt{k/m}$ is the natural frequency

$\zeta = c/c_{\text{critical}}$ is the damping ratio

$c_{\text{critical}} = 2\sqrt{k\,m}$ is the critical damping

$f_0 = F_0/m$ is the force magnitude per unit mass

The general solution of the inhomogeneous equation

$$\begin{cases} \ddot{u} + 2\zeta\,\omega_0 \dot{u} + \omega_0^2\, u = f_0 \cos(\omega t) \\ u(0) = x_0 \\ \dot{u}(0) = v_0 \end{cases}$$

Equation of motion and initial conditions

$$u(t) = u_{\text{homogeneous}}^{\text{general}}(t) + u_{\text{inhomogeneous}}^{\text{particular}}(t)$$

$$u_{\text{homogeneous}}^{\text{general}}(t) = \exp(-\zeta\omega_0 t)\big(A\cos(\omega_1 t) + B\sin(\omega_1 t)\big),$$

$$\omega_1 = \omega_0\sqrt{1-\zeta^2}$$

$$u_{\text{inhomogeneous}}^{\text{particular}}(t) = ?$$

A particular solution in terms of real-valued functions

$$\ddot{u} + 2\zeta\,\omega_0 \dot{u} + \omega_0^2\, u = f_0 \cos(\omega t)$$

$$u_{\text{inhomogeneous}}^{\text{particular}}(t) = U\cos(\omega t) \quad \Rightarrow \quad \left(-\omega^2 + \omega_0^2\right) U\cos(\omega t) - 2\zeta\omega\omega_0 U\sin(\omega t) = f_0\cos(\omega t)$$

$$u_{\text{inhomogeneous}}^{\text{particular}}(t) = U_c\cos(\omega t) + U_s\sin(\omega t)$$

$$\left(-\omega^2 U_c + 2\zeta\,\omega_0\omega U_s + \omega_0^2 U_c - f_0\right)\cos(\omega t) + \left(-\omega^2 U_s - 2\zeta\,\omega_0\omega U_c + \omega_0^2 U_s\right)\sin(\omega t) = 0$$

$$-\omega^2 U_c + 2\zeta\,\omega_0\omega U_s + \omega_0^2 U_c = f_0$$

$$-\omega^2 U_s - 2\zeta\,\omega_0\omega U_c + \omega_0^2 U_s = 0$$

$$\Rightarrow$$

$$U_c = f_0 \frac{\left(\omega_0^2 - \omega^2\right)}{\left(\omega_0^2 - \omega^2\right)^2 + 4\zeta^2\omega_0^2\omega^2}$$

$$U_s = f_0 \frac{2\zeta\,\omega_0\omega}{\left(\omega_0^2 - \omega^2\right)^2 + 4\zeta^2\,\omega_0^2\omega^2}$$

CTB 2300 Dynamics of Systems
Faculty of Civil Engineering and Geosciences
2022 Delft, The Netherlands

TU Delft
Delft University of Technology

A particular solution in terms of complex-valued functions

$$\ddot{u} + 2\zeta\,\omega_0 \dot{u} + \omega_0^2\, u = f_0 \cos(\omega t)$$

$$u_{\text{inhomogeneous}}^{\text{particular}}(t) = \operatorname{Re}\left(\hat{U}\exp(\mathrm{i}\omega t)\right)$$

$$\Rightarrow \operatorname{Re}\left\{\left(-\omega^2 + 2\mathrm{i}\zeta\omega\omega_0 + \omega_0^2\right)\hat{U}\exp(\mathrm{i}\omega t) - f_0\exp(\mathrm{i}\omega t)\right\} = 0$$

$$\Downarrow$$

$$\hat{U} = \frac{f_0}{-\omega^2 + 2\mathrm{i}\zeta\omega\omega_0 + \omega_0^2}$$

$$u_{\text{inhomogeneous}}^{\text{particular}}(t) = \operatorname{Re}\left(\frac{f_0}{-\omega^2 + 2\mathrm{i}\zeta\omega\omega_0 + \omega_0^2}\exp(\mathrm{i}\omega t)\right)$$

$$= \operatorname{Re}\left(\frac{f_0}{-\omega^2 + 2\mathrm{i}\zeta\omega\omega_0 + \omega_0^2}\left(\cos(\omega t) + \mathrm{i}\sin(\omega t)\right)\right)$$

$$= \frac{f_0}{\left(\omega_0^2 - \omega^2\right)^2 + 4\zeta^2\omega^2\omega_0^2}\left(\left(\omega_0^2 - \omega^2\right)\cos(\omega t) + 2\zeta\omega\omega_0\sin(\omega t)\right)$$

CTB 2300 Dynamics of Systems
Faculty of Civil Engineering and Geosciences
2022 Delft, The Netherlands

TU Delft
Delft University of Technology

Physics of the general solution

$$\ddot{u} + 2\zeta\,\omega_0 \dot{u} + \omega_0^2\, u = f_0 \cos(\omega t)$$

$$u(t) = \exp(-\zeta\omega_0 t)\left(A\cos(\omega_1 t) + B\sin(\omega_1 t)\right) + U_c\cos(\omega t) + U_s\sin(\omega t)$$

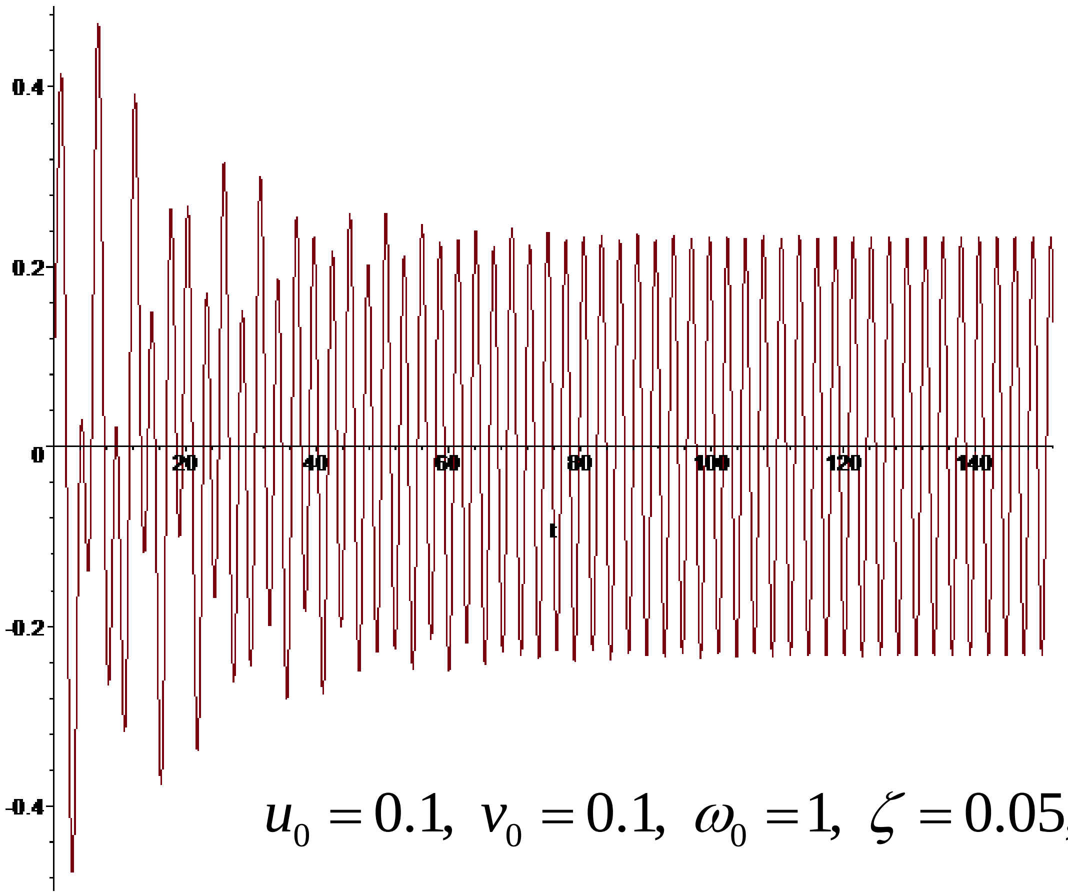
Damped Free Vibrations

Damped Forced Vibrations

The free vibration gradually subside with time!

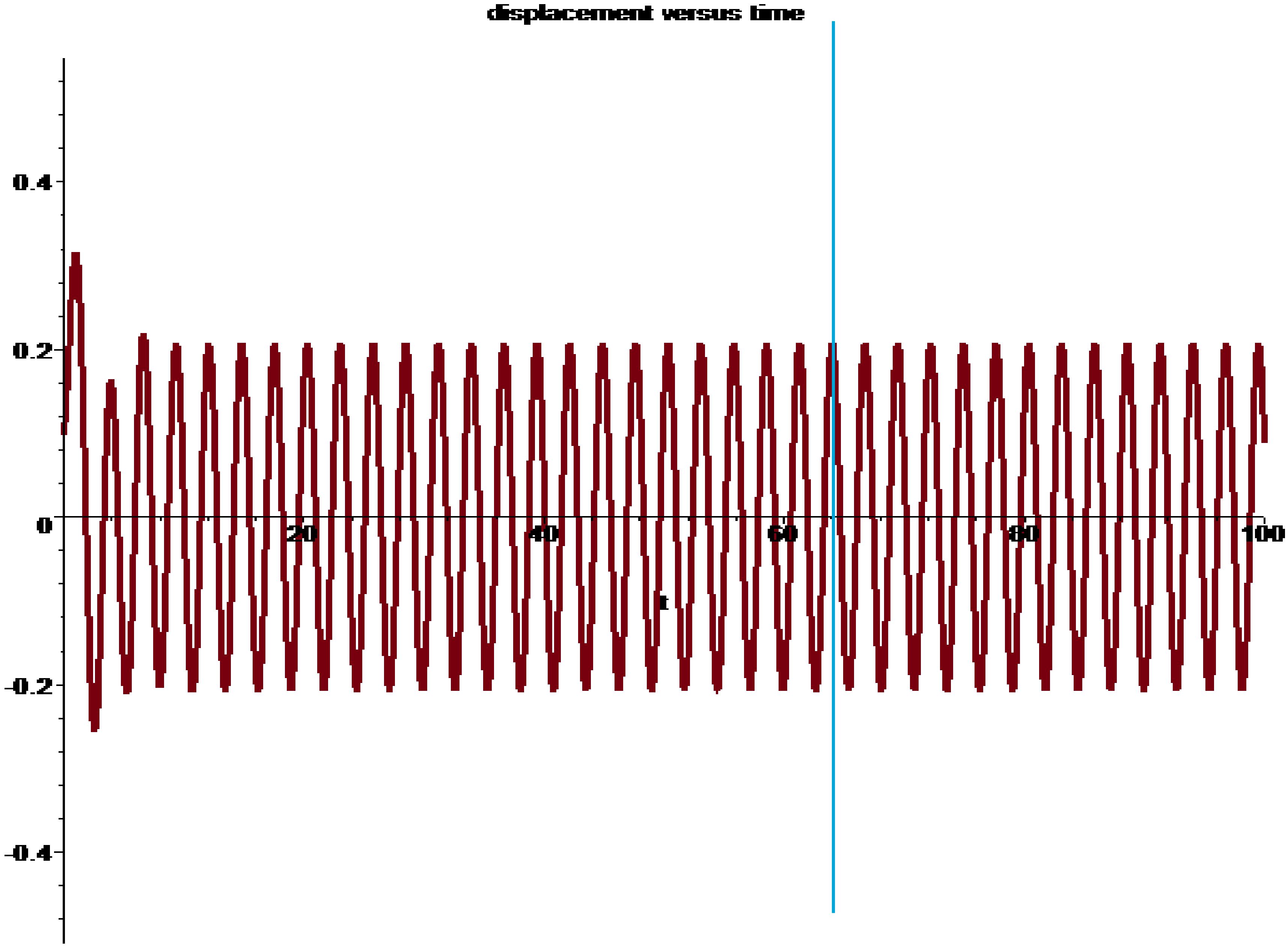
$$u_{\text{steady}}(t) = \lim_{t\to\infty} u(t) = U_c\cos(\omega t) + U_s\sin(\omega t)$$

The general solution: example plot



$$u_0 = 0.1,\ v_0 = 0.1,\ \omega_0 = 1,\ \zeta = 0.05,\ f_0 = 1,\ \omega = 2.3$$

The solution convergence to the steady-state



The steady-state response

$$u_{\text{steady}}(t) = \lim_{t\to\infty} u(t) = U_c \cos(\omega t) + U_s \sin(\omega t)$$

$$\Downarrow$$

$$u_{\text{steady}}(t) = U_{\text{steady}} \cos(\omega t - \varphi)$$

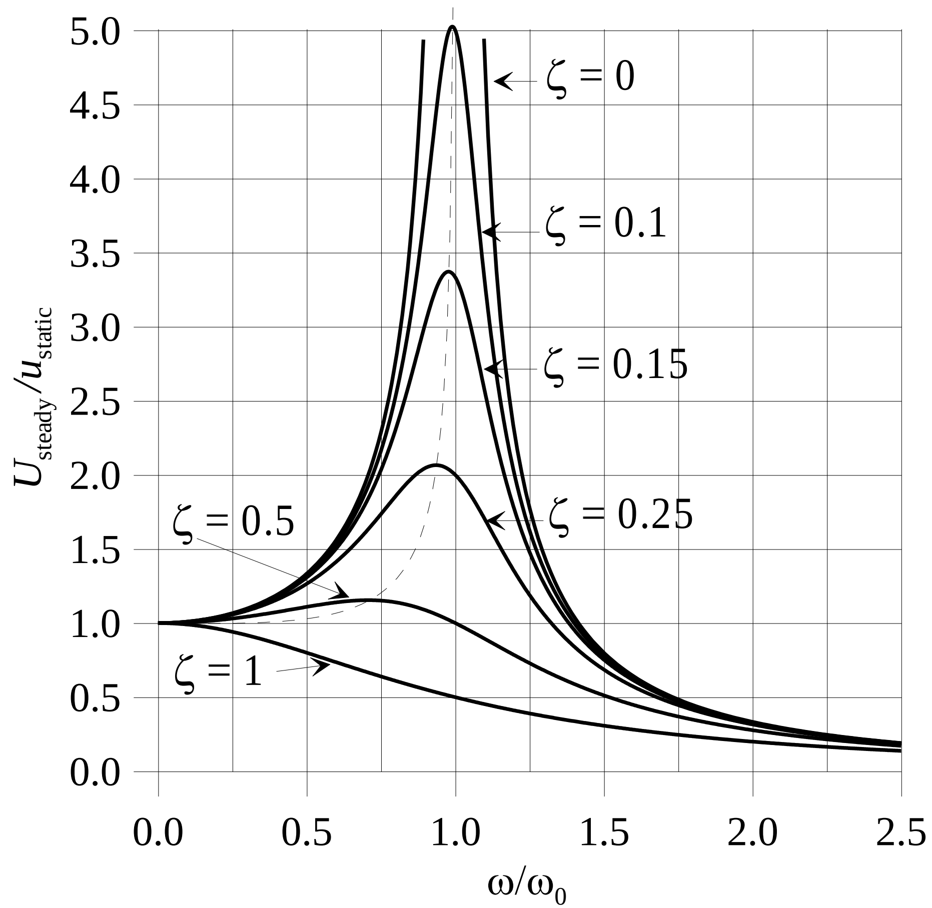
$$U_{\text{steady}} = \sqrt{U_c^2 + U_s^2} = \frac{u_{\text{static}}}{\sqrt{\left(1-\omega^2/\omega_0^2\right)^2 + 4\zeta^2\,\omega^2/\omega_0^2}} \quad - \quad \text{amplitude}$$

$$\varphi = \text{Arctan}\left(\frac{U_s}{U_c}\right) = \text{Arctan}\left(\frac{2\zeta\,\omega/\omega_0}{1-\omega^2/\omega_0^2}\right) \quad - \quad \text{phase lag}$$

The magnification factor (also called "dynamic amplification factor" and "amplitude-frequency characteristic")

$$u_{\text{steady}}(t) = U_{\text{steady}} \cos(\omega t - \varphi)$$

$$\frac{U_{\text{steady}}}{u_{\text{static}}} = \frac{1}{\sqrt{\left(1-\omega^2/\omega_0^2\right)^2 + 4\zeta^2\,\omega^2/\omega_0^2}}$$



Video: Experiment SDOF

The phase lag
(also called the “phase-frequency characteristic”)

$$u_{\text{steady}}(t) = U_{\text{steady}} \cos(\omega t - \varphi)$$

$$\varphi = \text{Arctan}\left(\frac{2\zeta\, \omega/\omega_0}{1 - \omega^2/\omega_0^2}\right)$$

Phase lag