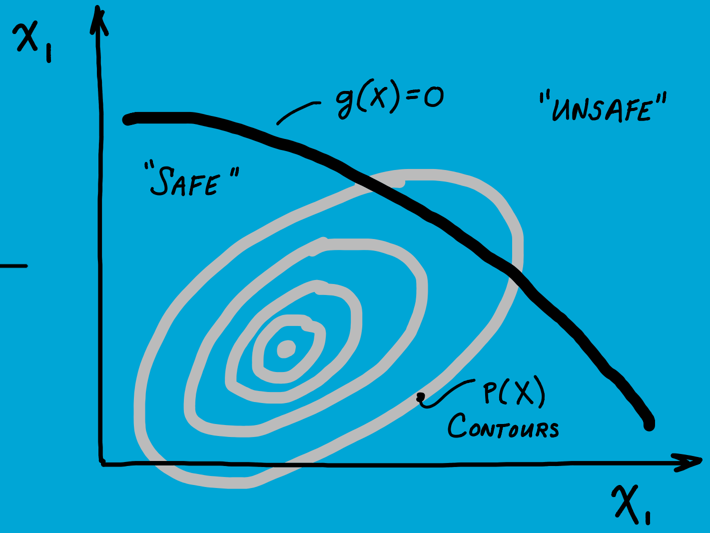
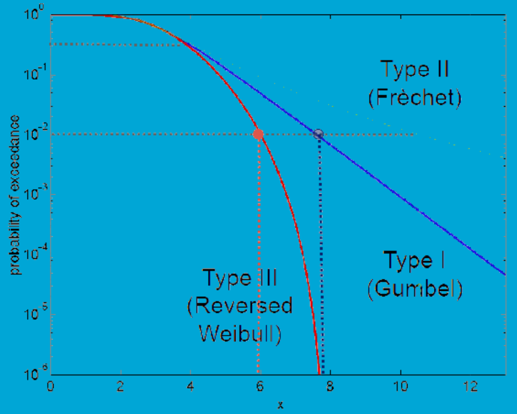
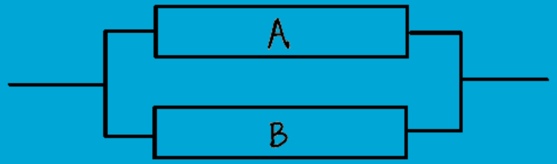
CIEM42X0 Probabilistic Design

Hydraulic and Offshore Structures (HOS) Track

Civil Engineering MSc Program

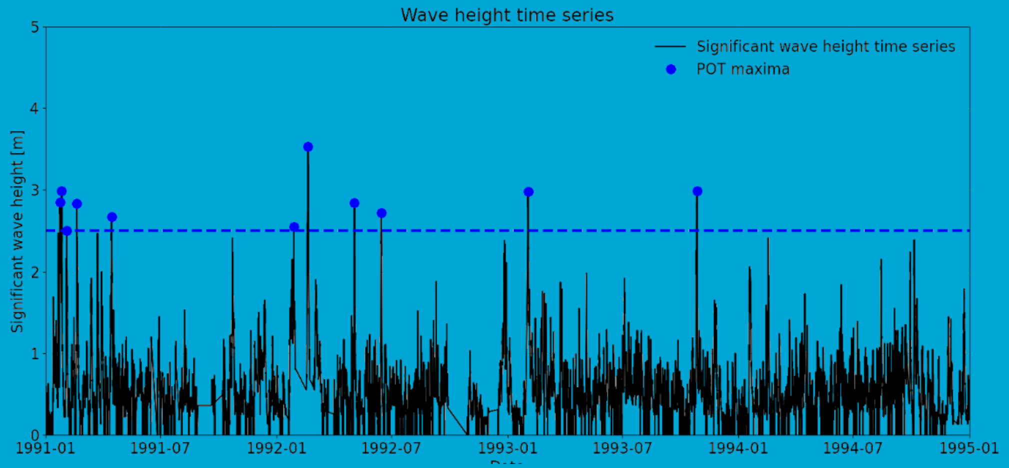






Extreme Value Analysis: basics

Patricia Mares Nasarre



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What have you seen so far?

1. Identify what is an **extreme value** and apply it within the engineering context

2. Interpret and apply the concept of **return period and design life**

3. Apply **extreme value analysis** to datasets

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Learning objectives

1. Identify what is an **extreme value** and apply it within the engineering context

2. Interpret and apply the concept of **return period and design life**

3. Apply **extreme value analysis** to datasets

4. Apply techniques to **support the threshold selection**

Join the Vevox session

Go to **vevox.app**

Enter the session ID: **178-023-242**

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What is an extreme in probability theory?

16

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Showing Results

What is an extreme in probability theory?

outlier

far from average

value far from mean

is an event

a maximum

the maximum value

depends on method

a peak

value above threshol

block max

independent

an very high or low

very big values

over threshold

highest/lowest value

Extremes and Extreme Value Analysis

An **extreme observation** is an observation that **deviates from the average observations.**

Infrastructures and systems are designed to **withstand extreme conditions (ULS)**.

- Breakwater → wave storm
- Flood defences → floods, droughts

To properly design and assess infrastructures and system **we need to characterize the uncertainty of the loads**.

Extreme Value Analysis

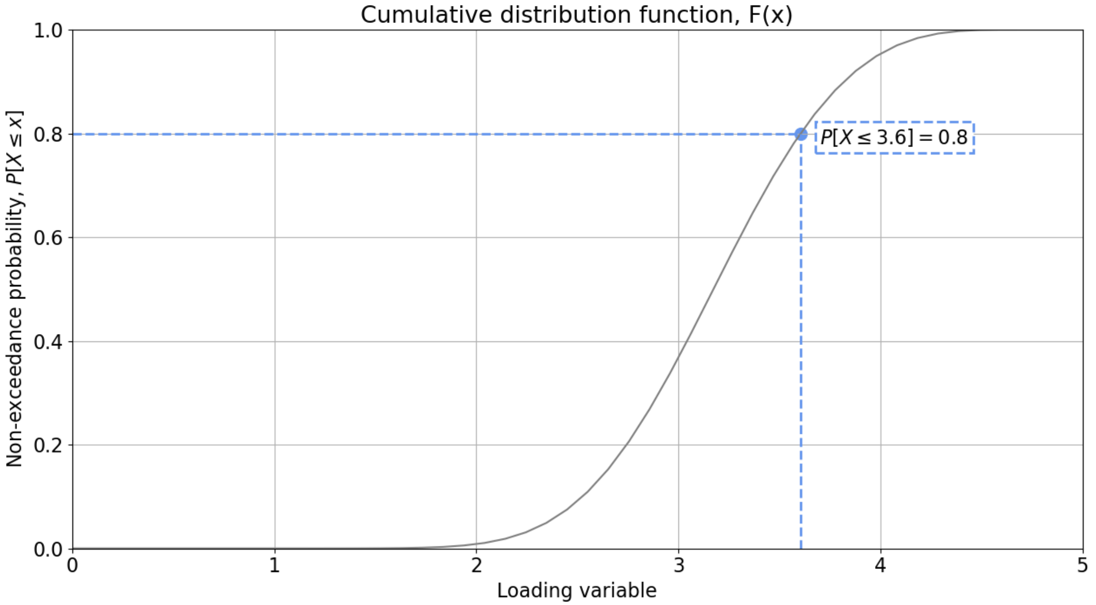
Based on historical observed extremes (limited)…

- Allows us to **model** the stochastic behaviour of extreme events

- Allows us to **infer** extremes we have not observed yet (extrapolation)

Time series of **observations** of the loading variable

EVA



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Imagine you are working with a continuous variable, such as the discharge in a river (Q). You want to use the cumulative distribution function (CDF) to compute the probability of observing a discharge Q=100 m3/s. Which probability would you obtain?

Exceedance probability, P[Q>100]	0%
Non-exceedance probability, P[Q<100]	0%
Probability of the event, P[Q=100]	0%

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Imagine you are working with a continuous variable, such as the discharge in a river (Q). You want to use the cumulative distribution function (CDF) to compute the probability of observing a discharge Q=100 m3/s. Which p

Exceedance probability, P[Q>100] — 35%

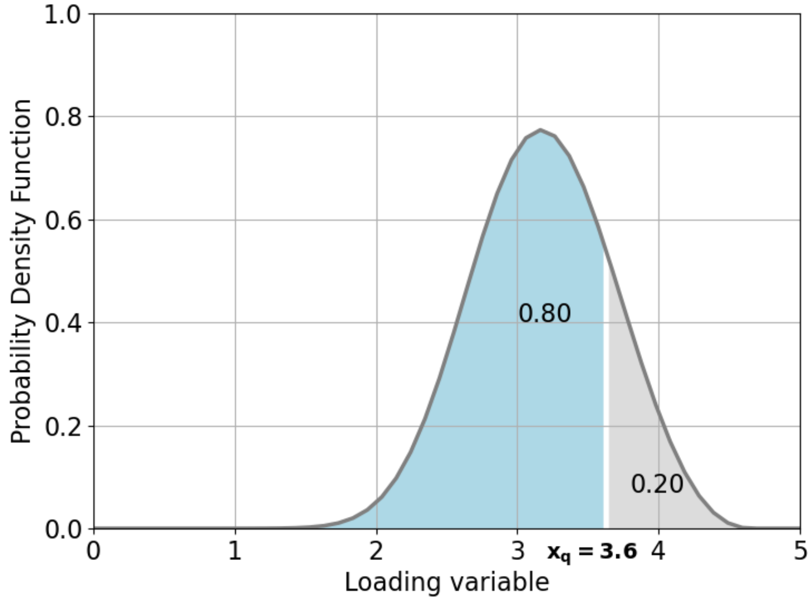
Non-exceedance probability, P[Q<100] — 45%

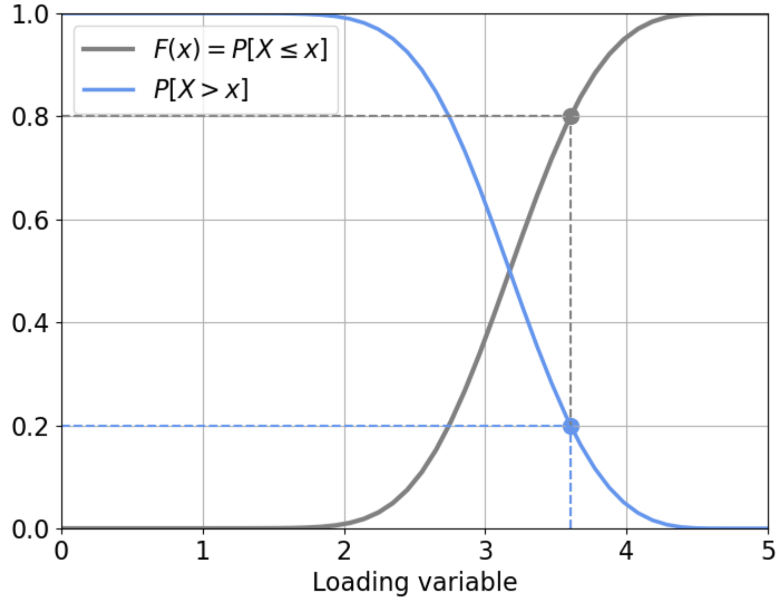
Probability of the event, P[Q=100] — 20%

Percentile and Exceedance Probability

Consider x_q such that $\mathbf{Pr}(X \leq x_q) = F(x_q) = q$

- x_q is the **qth – percentile**
- $\mathbf{Pr}(X > x_q) = 1 - F(x_q) = \mathbf{1} - q = p$ is the **exceedance probability**





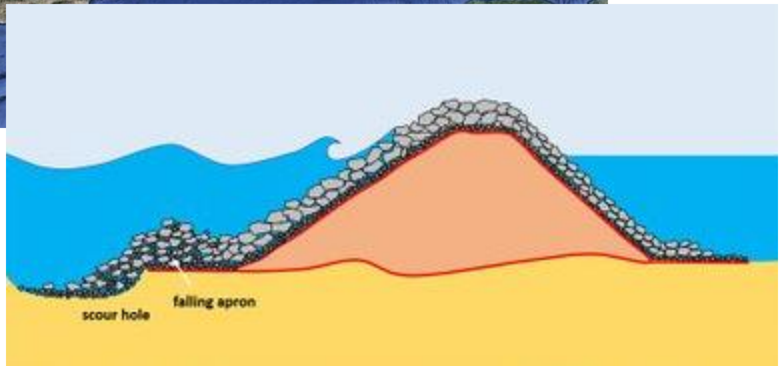
80th-percentile: $x_q = 3.60$

$Pr(X \leq 3.6) = 0.8$

Exceedance probability

$Pr(X > x_q) = 0.20$

Example case: intervention in the Mediterranean coast



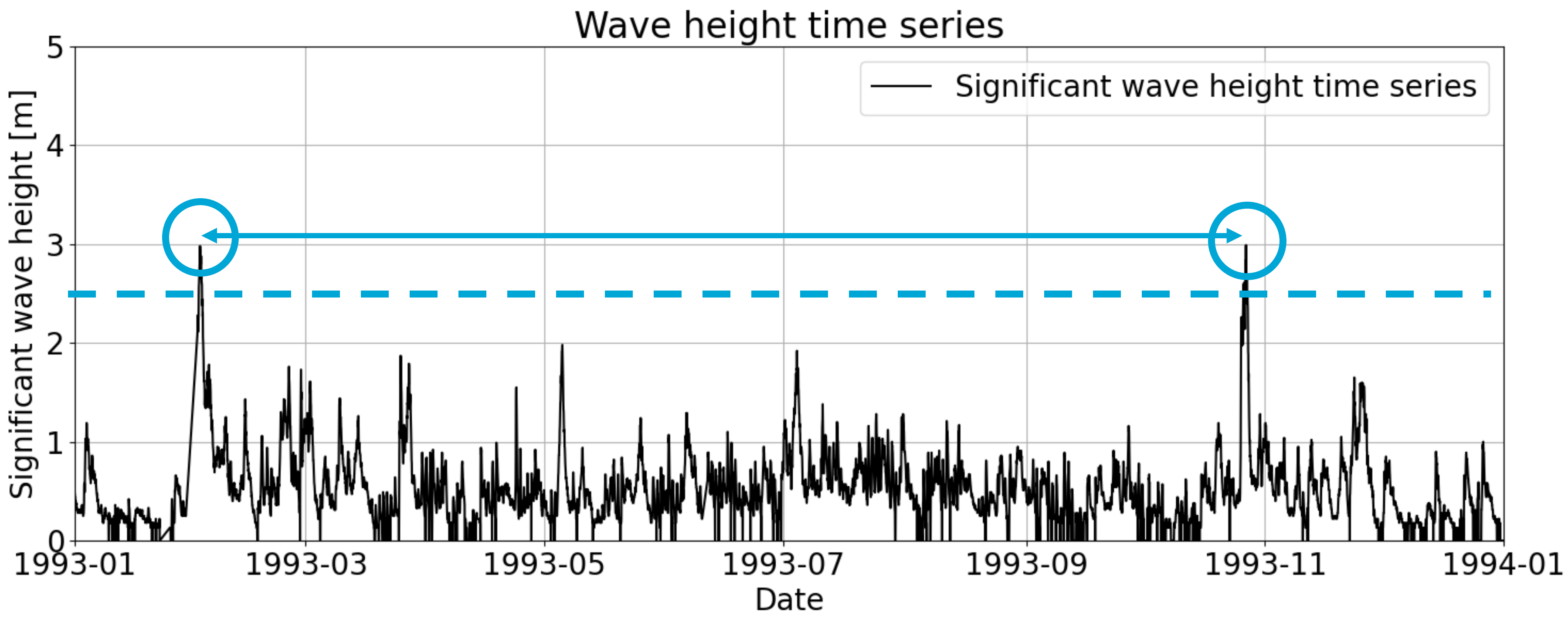
- It may be a coastal structure, a water intake, the restoration of a sandy beach, between others.
- Here: **design a mound breakwater**
- Mound breakwater must resist wave storms → H_s
- ***But which one?***

Return Period

The Return Period (T_R) is the expected time between exceedances. "In other words, we have to make, on average, $1/p_{f,y}$ trials in order that the event happens once" (Gumbel) or **wait $1/p_{f,y}$ years before the next occurrence**, being $p_{f,y}$ the exceedance probability.

Assumption of stationarity:
Every year the probability of the event being higher/lower than the threshold is always the same

$$T_R(t) = \frac{1}{p_{f,y}}$$



Design requirements – Binomial distribution

$$\boxed{T_R = \frac{1}{\mathrm{p}_{\mathrm{f,y}}} = \frac{1}{1-(1-\mathrm{p}_{\mathrm{f,DL}})^{1/DL}}}$$

- DL = 20 years
- $\mathrm{p}_{\mathrm{f,DL}}$ = 0.20

$$T_R = \frac{1}{\mathrm{p}_{\mathrm{f,y}}} = \frac{1}{1-(1-0.2)^{1/20}} \approx 90\ years$$

$$\mathrm{p}_{\mathrm{f,y}} \approx 0.011$$

Learning objectives

1. Identify what is an **extreme value** and apply it within the engineering context
2. Interpret and apply the concept of **return period and design life**
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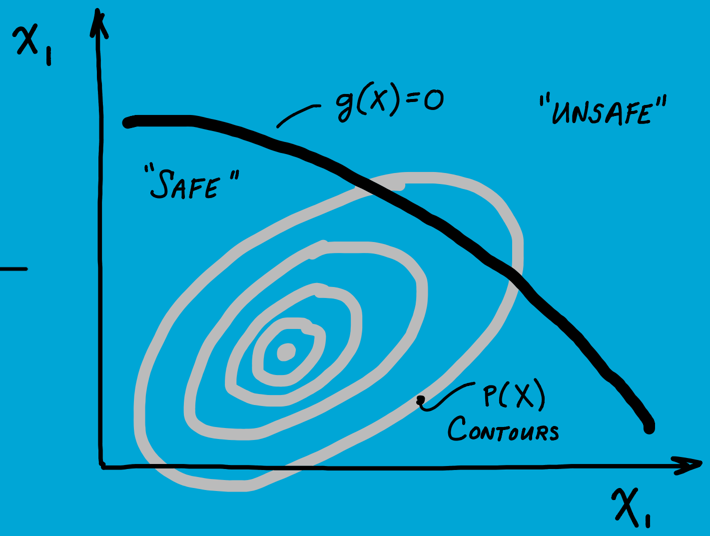
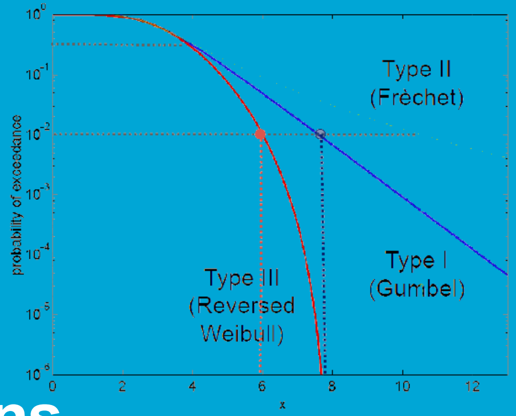
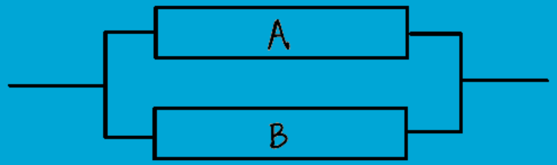
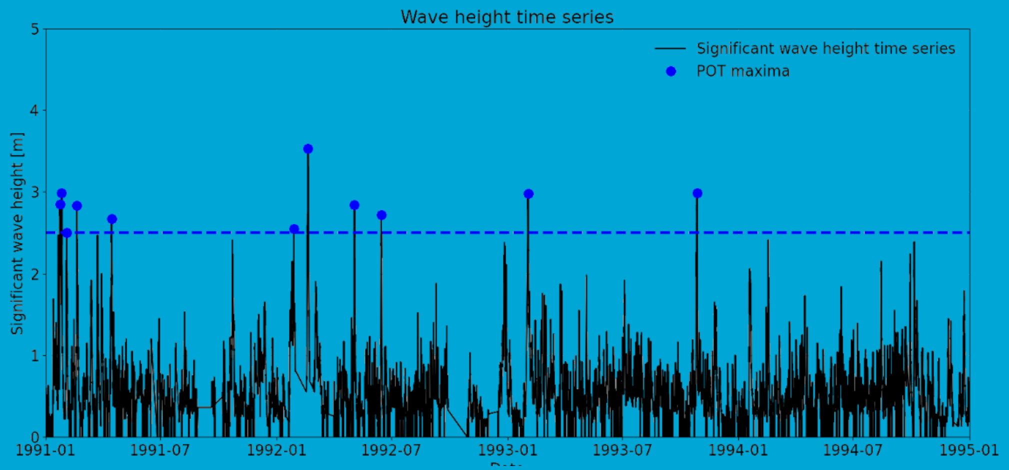






EVA: Sampling and distributions



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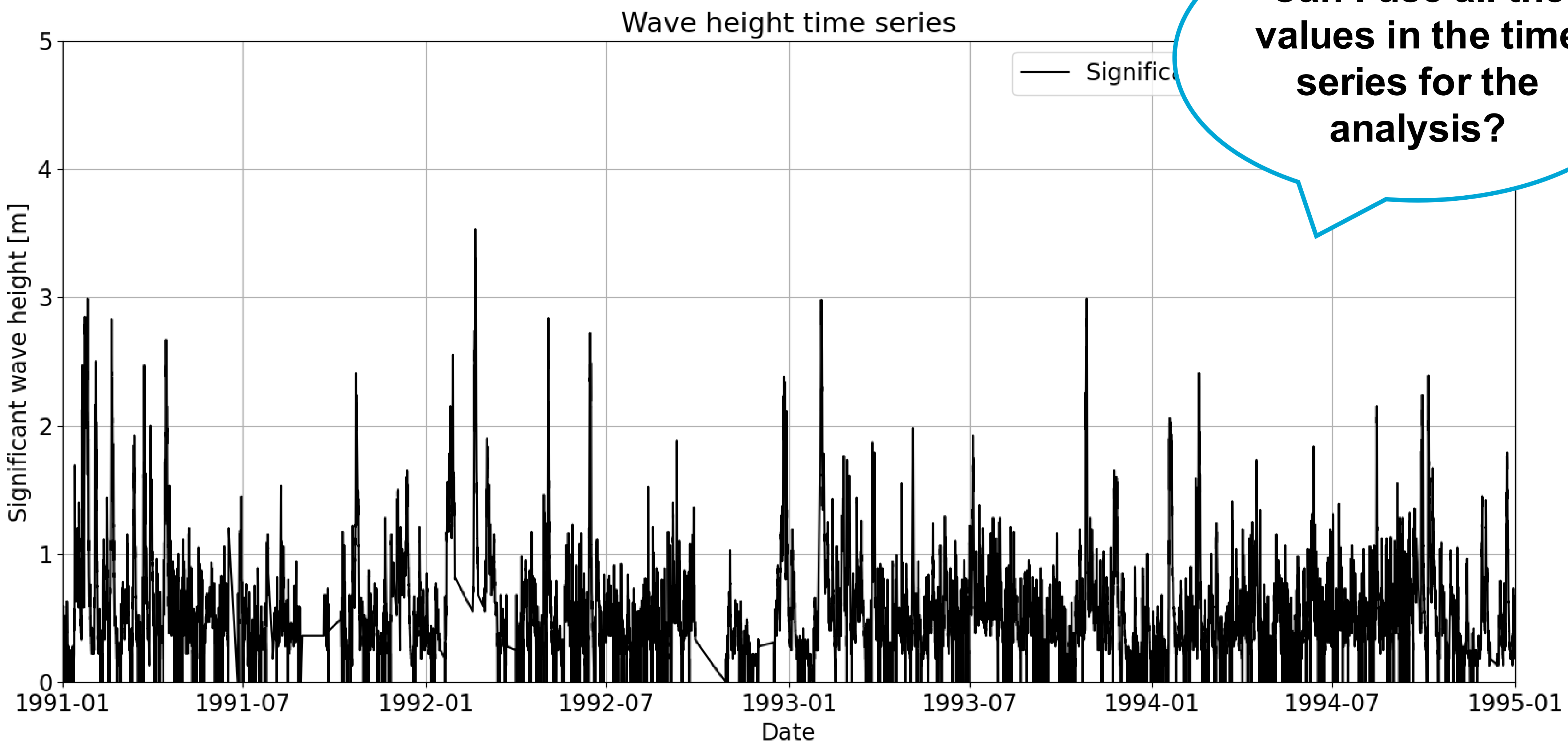
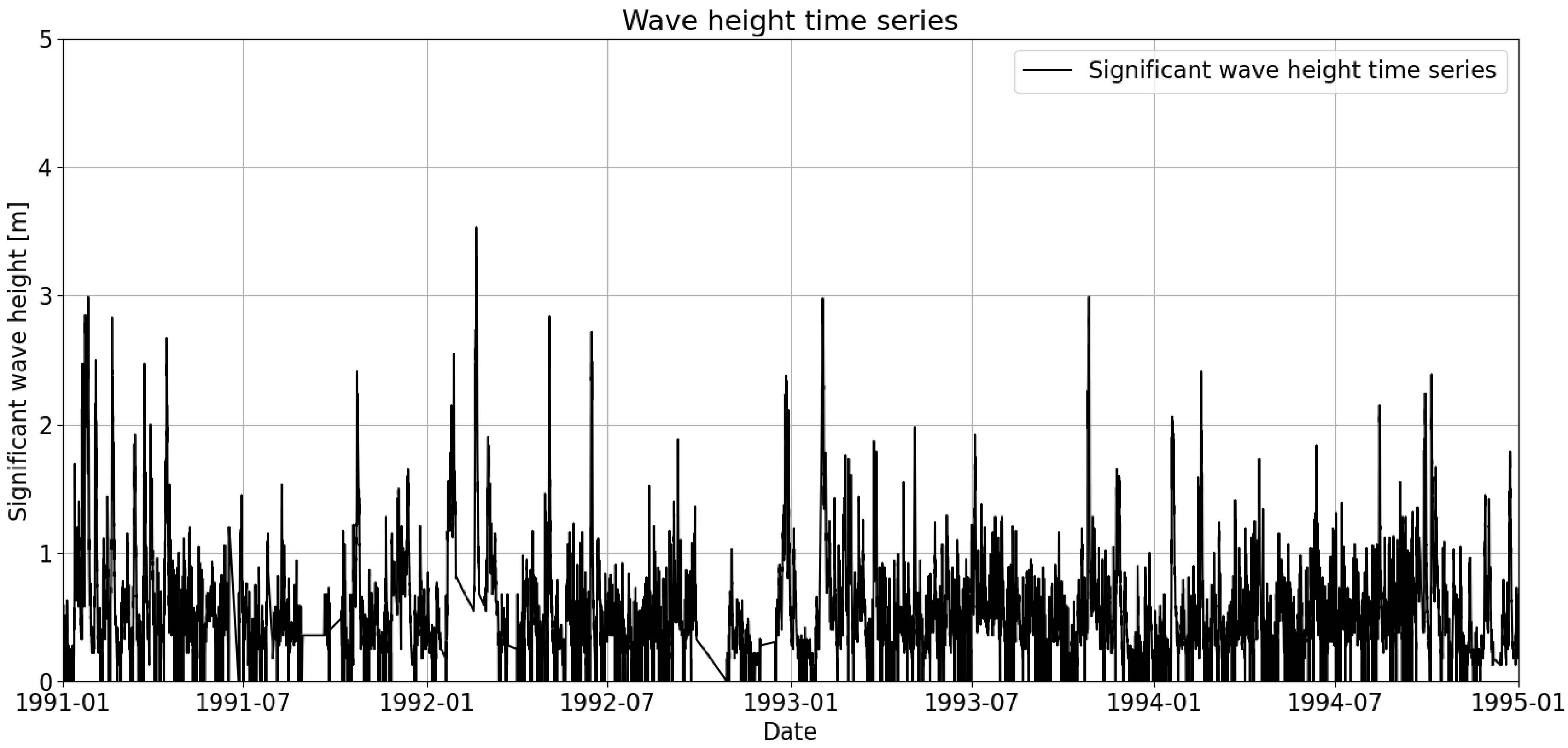
Time series



Can I use all the values in the time series for the analysis?

We need to sample extreme values!



Which one of the following options is a sampling technique for extremes? You may select more than one option.

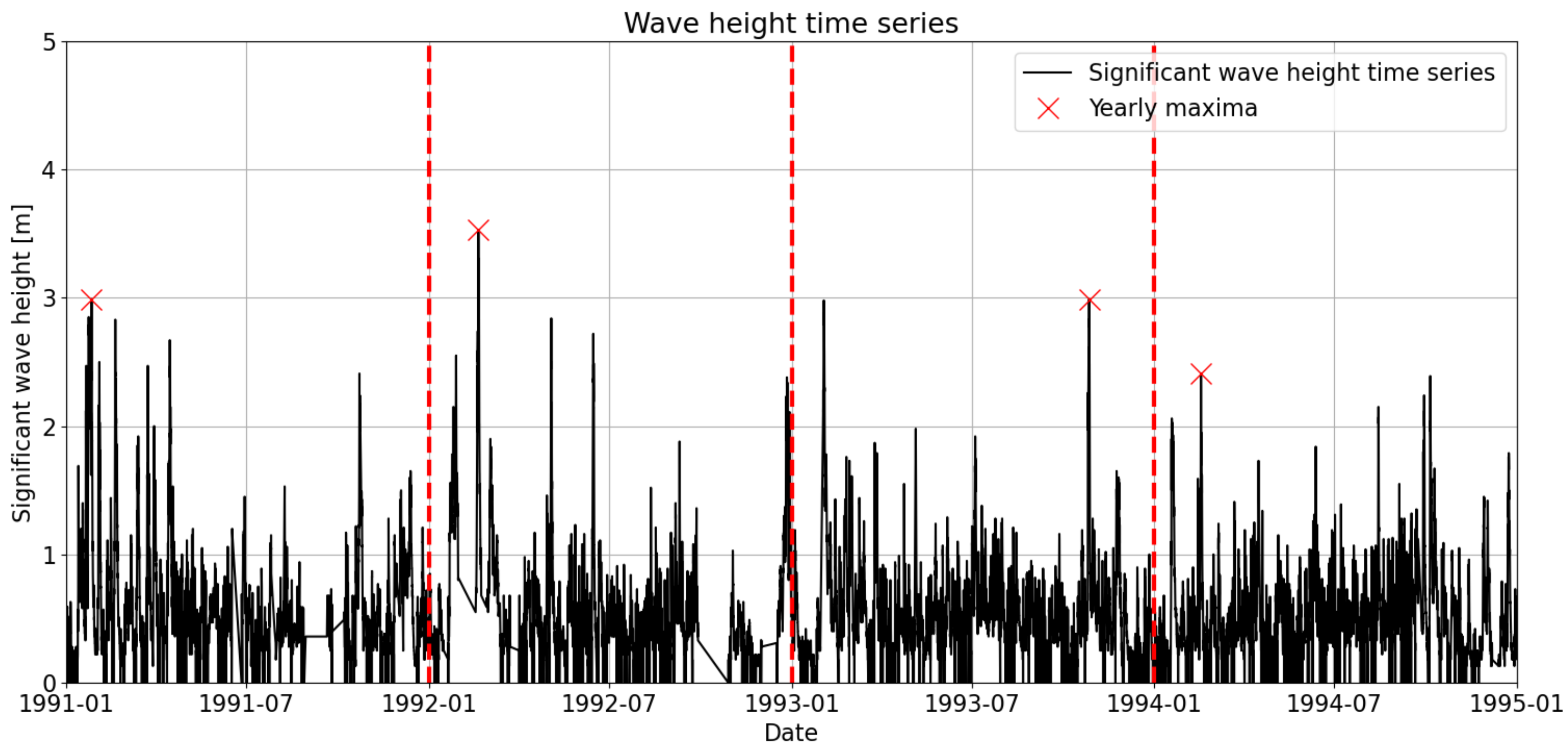
Peak Over Threshold	0%
Block Selection	0%
Generalized Extreme Value (GEV)	0%
Point Over Threshold	0%
Block Maxima	0%

Which one of the following options is a sampling technique for extremes? You may select more than one option.

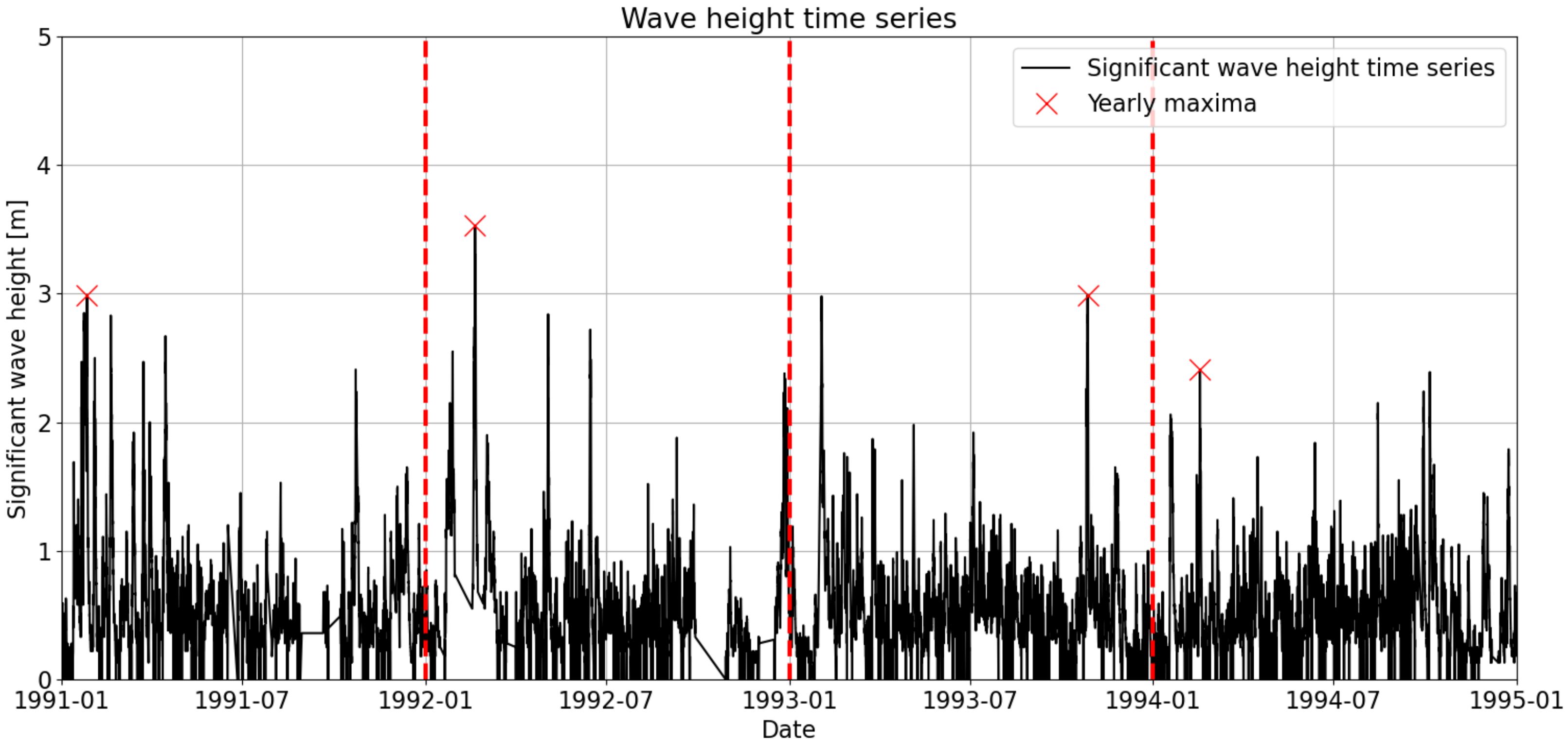
Peak Over Threshold	100%
Block Selection	0%
Generalized Extreme Value (GEV)	0%
Point Over Threshold	0%
Block Maxima	95.45%

Sampling extremes: Block Maxima

1. Block Maxima



Sampling extremes: Block Maxima



1. **Block Maxima** (typically block=1year)
    - Maximum value within the block
    - Number of selected events=number of blocks recorded (e.g.: number of years)
    - Easy to implement

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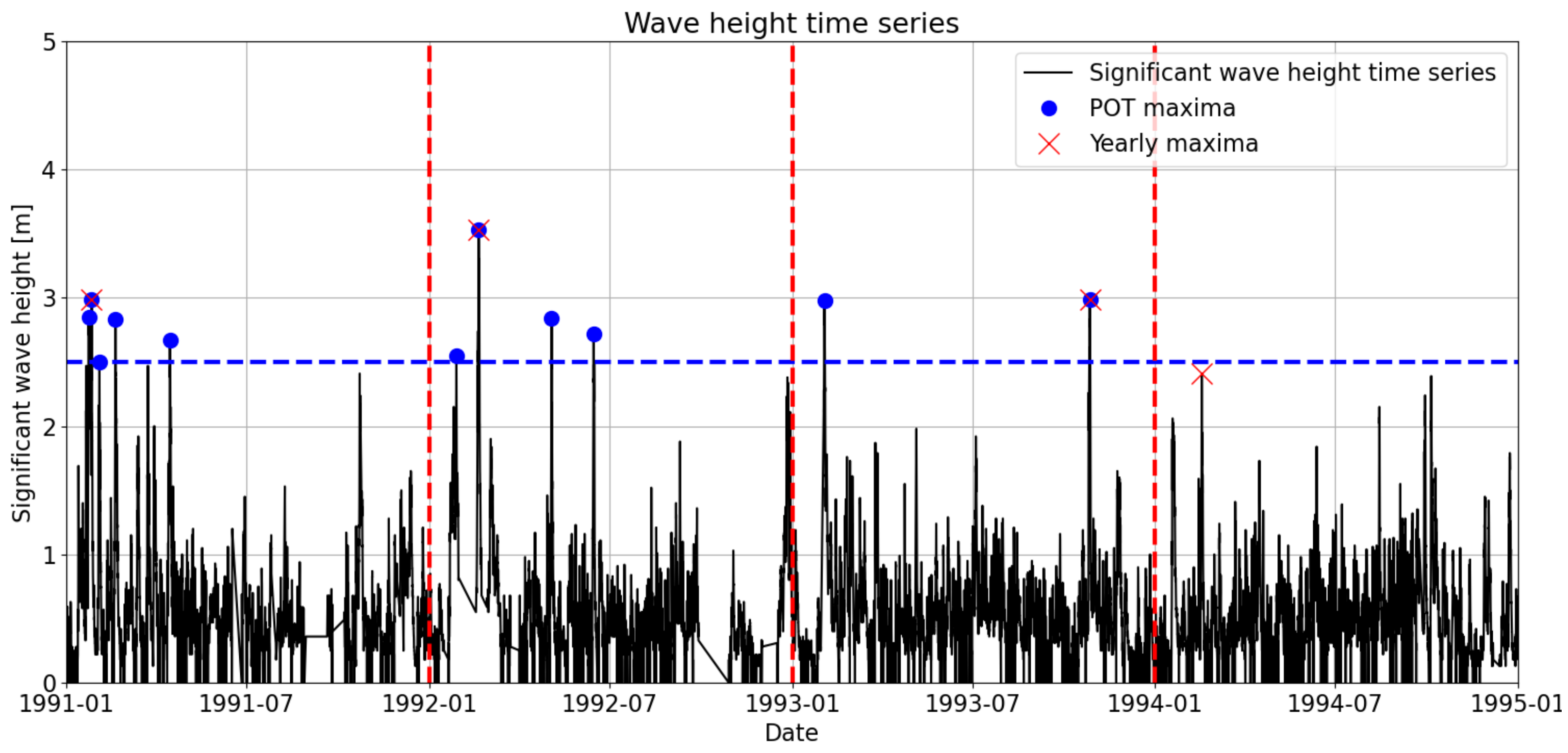
Sampling extremes: Peak Over Threshold (POT)



2. Peak Over Threshold (POT)

- Usually, higher number of extremes identified
- Additional parameters:
  - Threshold (*th*)
  - Declustering time (*dl*)

And what about the distributions?

Choose the right pairs of sampling technique with distribution function.

Peak Over Threshold (POT) with Generalized Pareto Distribution (GPD) — 0%

Block Maxima (BM) with Generalized Pareto Distribution (GPD) — 0%

Block Maxima (BM) with Generalized Extreme Value distribution (GEV) — 0%

Peak Over Threshold (POT) with Generalized Extreme Value distribution (GEV) — 0%

Choose the right pairs of sampling technique with distribution function.

Peak Over Threshold (POT) with Generalized Pareto Distribution (GPD)	63.16%
Block Maxima (BM) with Generalized Pareto Distribution (GPD)	31.58%
Block Maxima (BM) with Generalized Extreme Value distribution (GEV)	63.16%
Peak Over Threshold (POT) with Generalized Extreme Value distribution (GEV)	36.84%

Block Maxima and Generalized Extreme Value Distribution

- We are interested in modelling the maximum of the sequence $X = X_1, \dots, X_n$ of *iid* random variables, $M_n = \max(X_1, \dots, X_n)$, where *n* is the number of observations in a given block.

- We can prove that for large *n*, **those maxima tend to the Generalized Extreme Value (GEV) family of distributions, regardless the distribution of *X*.**

$$P[M_n \leq x] \rightarrow G(x)$$

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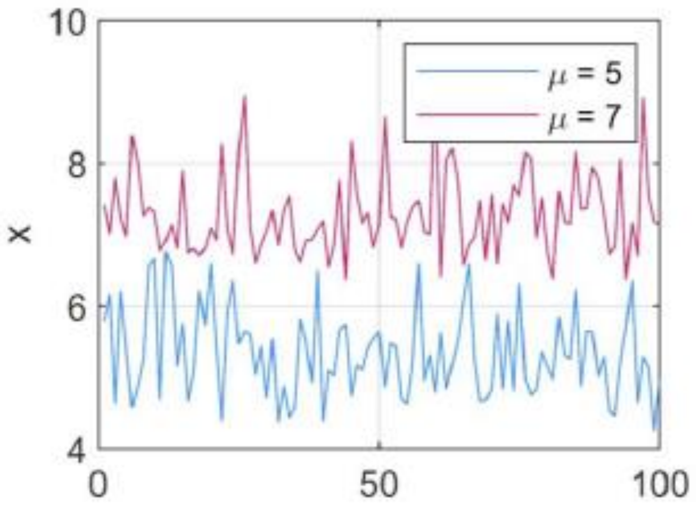
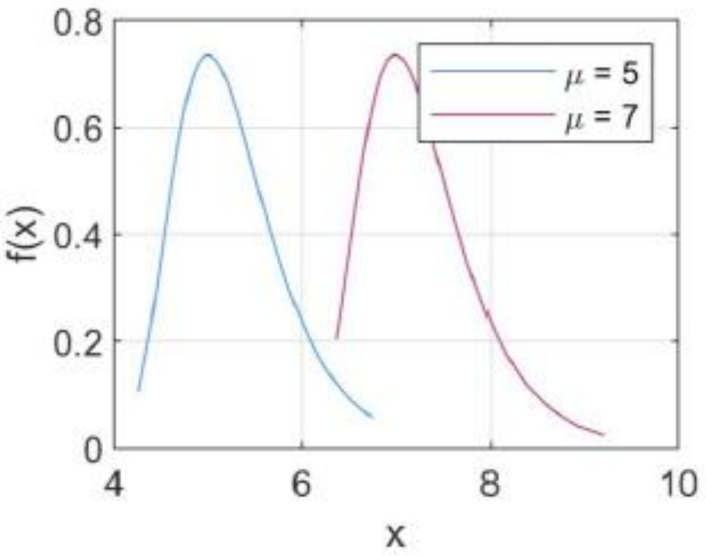
Block Maxima and Generalized Extreme Value Distribution

Generalized Extreme Value is defined as

$$G(x) = exp - [1 + \xi \frac{x-\mu}{\sigma}]^{-1/\xi} \qquad (1 + \xi \frac{x-\mu}{\sigma}) > 0$$

With parameters location ($-\infty < \mu < \infty$), scale ($\sigma > 0$) and shape ($-\infty < \xi < \infty$).





Location parameter (μ)

Higher μ, right displacement of the distribution, higher values.

Block Maxima and Generalized Extreme Value Distribution

Generalized Extreme Value is defined as

$$G(x) = exp-[1+\xi\frac{x-\mu}{\sigma}]^{-1/\xi} \qquad (1+\xi\frac{x-\mu}{\sigma}) > 0$$

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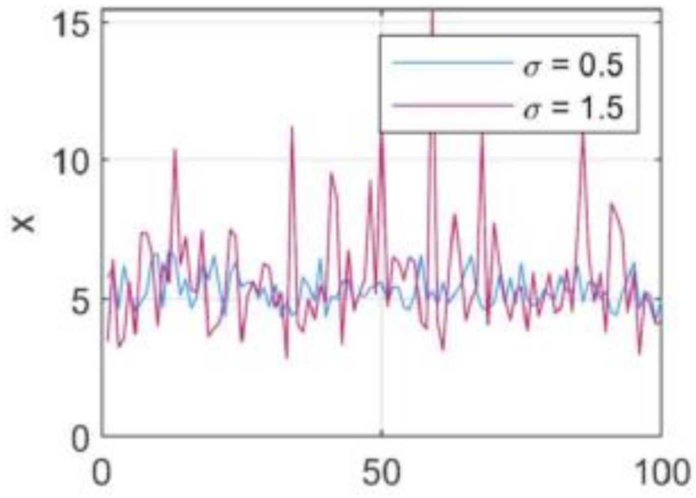

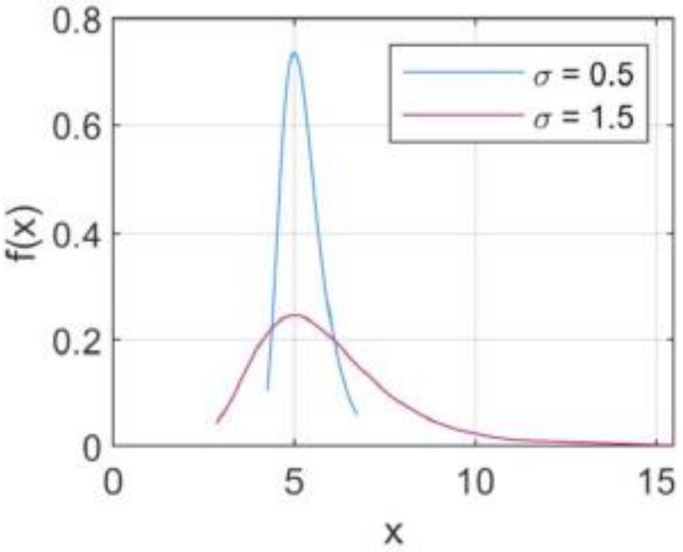

Scale parameter (σ)

Higher $\boldsymbol{\sigma}$, wider distribution.

Block Maxima and Generalized Extreme Value Distribution

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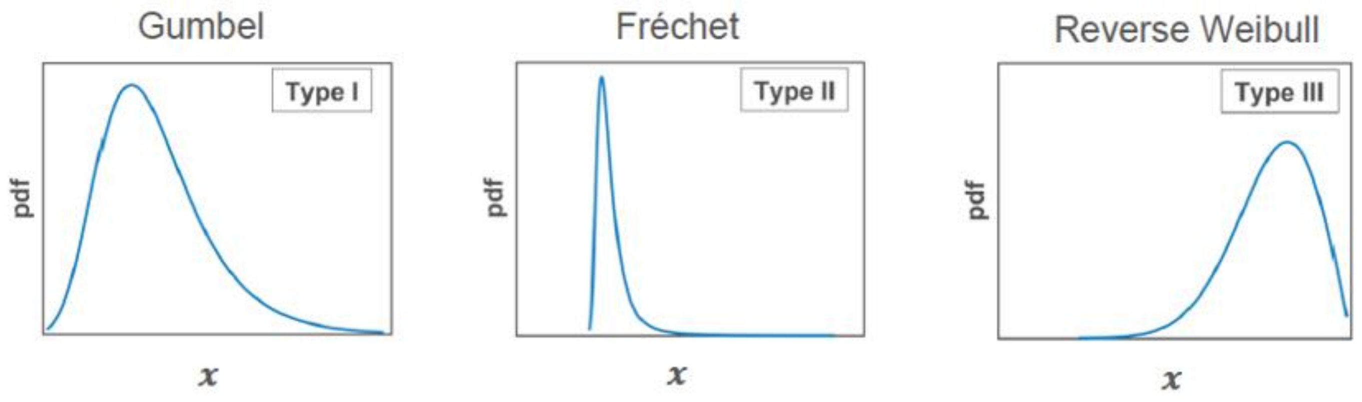


$\xi \to 0$ *Exponential decay*

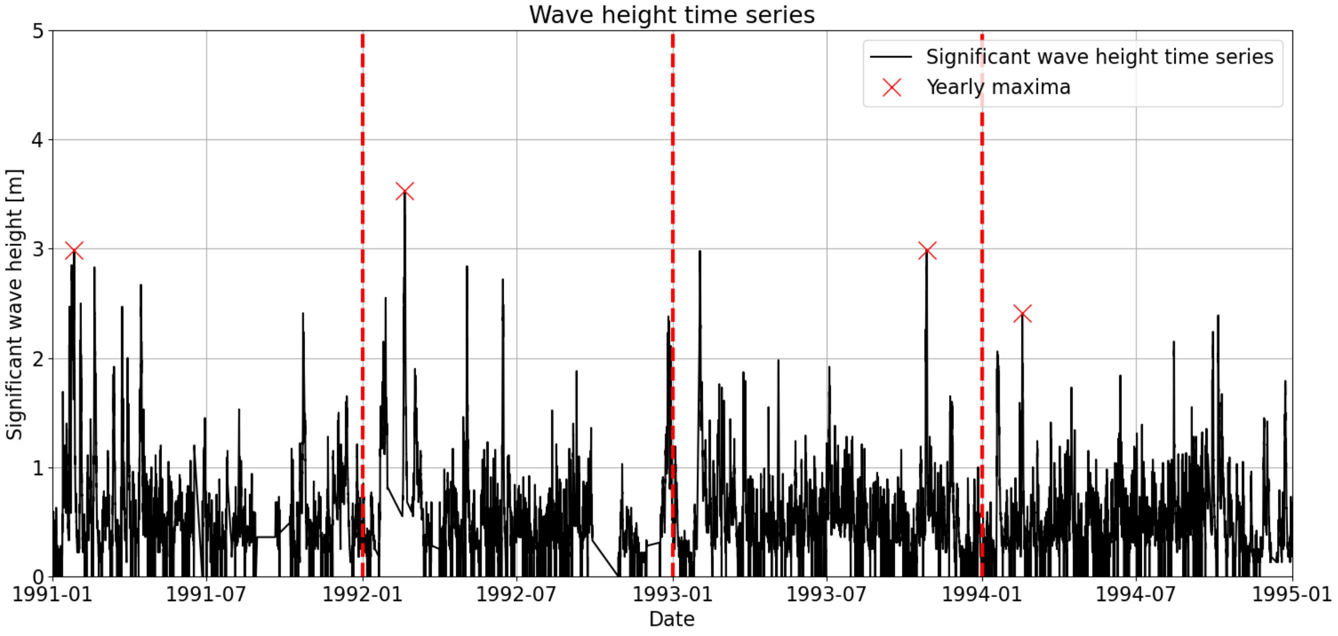
$\xi > 0$ *Polynomial decay*

$\xi < 0$ *Upper bound* $(\mu - {}^{\sigma}/_{\xi})$

Shape parameter (ξ)

Determines the tail of the distribution.

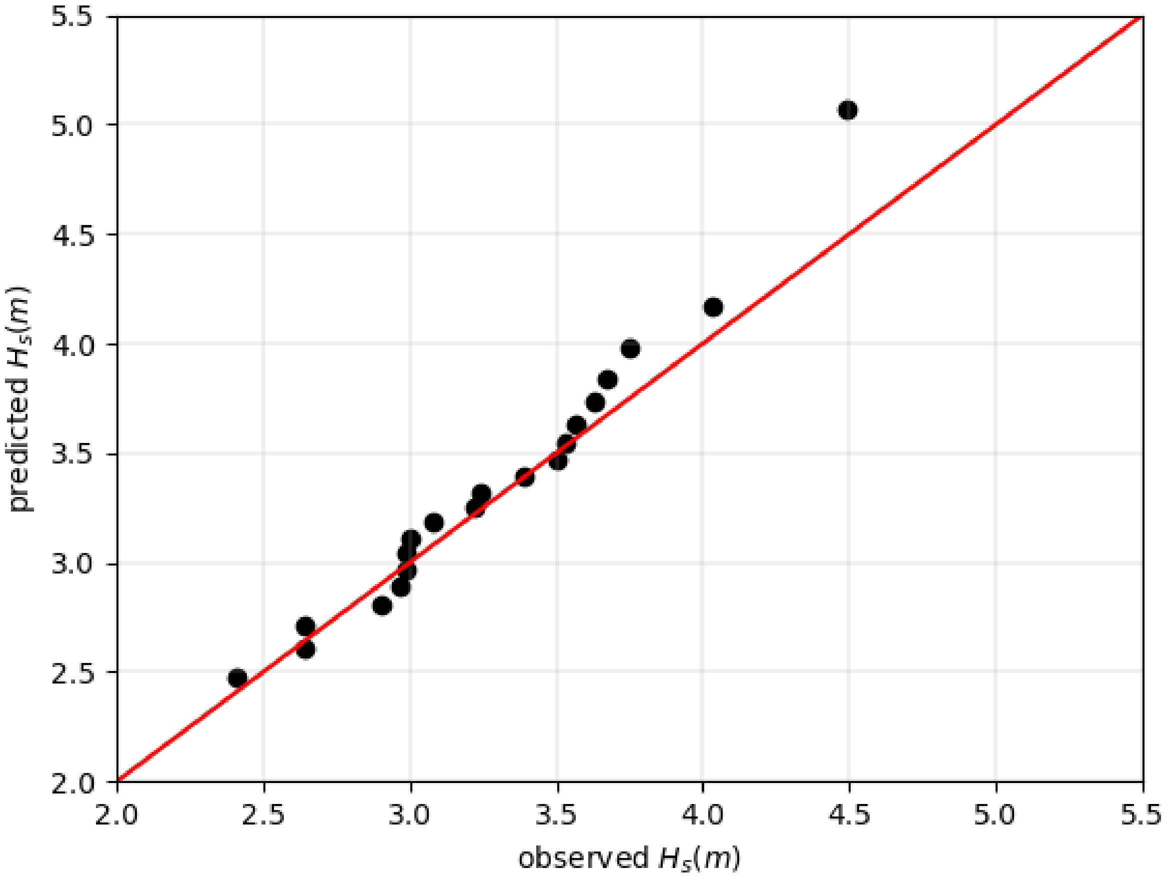
Let's apply it



- **Load: significant wave height (T_R=90 years)**
- 20 years of hourly measurements → **20 yearly maxima samples**

```
read observations

for each year i:
obs_max[i] = max(observations in year i)
end

fit GEV(obs_max)

check fit (e.g., QQ-plot or Kolmogorov-Smirnov test)

inverse GEV to determine the design event
```

Let's apply it



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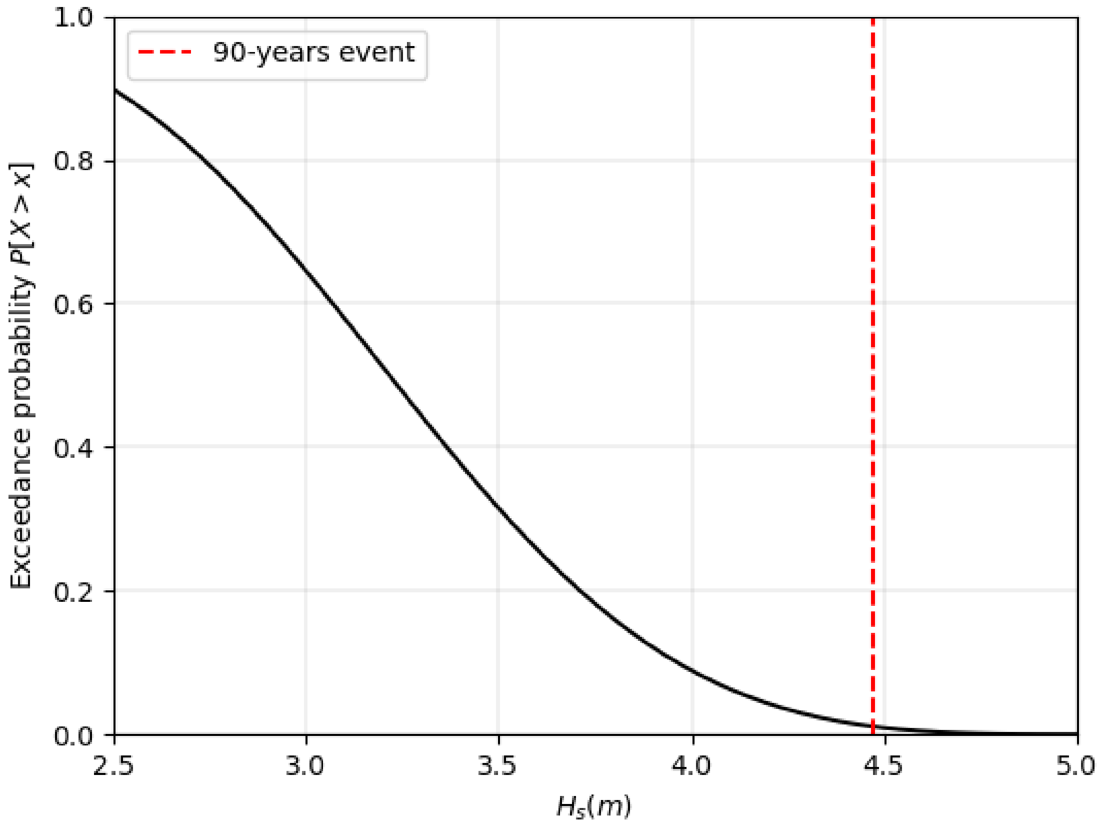
inverse GEV to determine the design event

Let's apply it

$$z_p = G^{-1}(1-p_{f,y}) = \begin{cases} \mu - \frac{\sigma}{\xi}[1-\{-log(1-p_{f,y})\}^{-\xi}] & for\ \xi \neq 0 \\ \mu - \sigma log\{1-p_{f,y}\} & for\ \xi = 0 \end{cases}$$



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inverse GEV to determine the design event

Common mistakes - Let's talk about the units

- **<u>Daily maxima</u>** of discharges Q is performed on the observations which last for 5 years. We have then 365x5=1,825 extremes. A GEV is fitted.

- We want to compute the discharge associated with a **<u>return period of 100 years</u>**.

??

$$z_p = G^{-1}(1 - p_{f,y}) = \begin{cases} \mu - \frac{\sigma}{\xi}[1 - \{-log(1 - \boxed{p_{f,y}})\}^{-\xi}] & for\ \xi \neq 0 \\ \mu - \sigma log\{1 - p_{f,y}\} & for\ \xi = 0 \end{cases}$$

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Common mistakes - Let’s talk about the units

- Daily maxima: ‘units’ of the probabilities in the GEV distribution?

Common mistakes - Let's talk about the units

- Daily maxima: 'units' of the probabilities in the GEV distribution $\frac{1}{days}$

- Return period: 100 years

$$T_R = \frac{1}{\mathrm{p}_{\mathrm{f,y}}} \rightarrow \mathrm{p}_{\mathrm{f,y}} = \frac{1}{T_R} = \frac{1}{100\ years}$$

$$T_R = \frac{1}{\mathrm{p}_{\mathrm{f,y}}} \rightarrow \mathrm{p}_{\mathrm{f,y}} = \frac{1}{T_R} = \frac{1}{100\ years}\frac{1\ year}{365\ days} = 2.7 \cdot 10^{-5}\ \text{1/days}$$

$$z_p = G^{-1}(1-p_{f,y}) = \begin{cases} \mu - \frac{\sigma}{\xi}[1-\{-log(1-\boxed{p_{f,y})}\}^{-\xi}] & for\ \xi \neq 0 \\ \mu - \sigma log\{1-p_{f,y}\} & for\ \xi = 0 \end{cases}$$

POT and Generalized Pareto Distribution

- The maximum of the sequence $X = X_1, \ldots, X_n$ of *iid* random variables, $M_n = \max(X_1, \ldots, X_n)$, where *n* is the number of observations in a given block, follows **the Generalized Extreme Value (GEV) family of distributions, regardless the distribution of *X*** for large *n*.

$$P[M_n \leq x] \rightarrow G(x)$$

- If that is true, **the distribution of the excesses can be approximated by a Generalized Pareto distribution**.

$$F_{th} = P[X - th \leq x | X > th] \rightarrow H(y)$$

- where the excesses are defined as $Y = X - th$ for $X > th$

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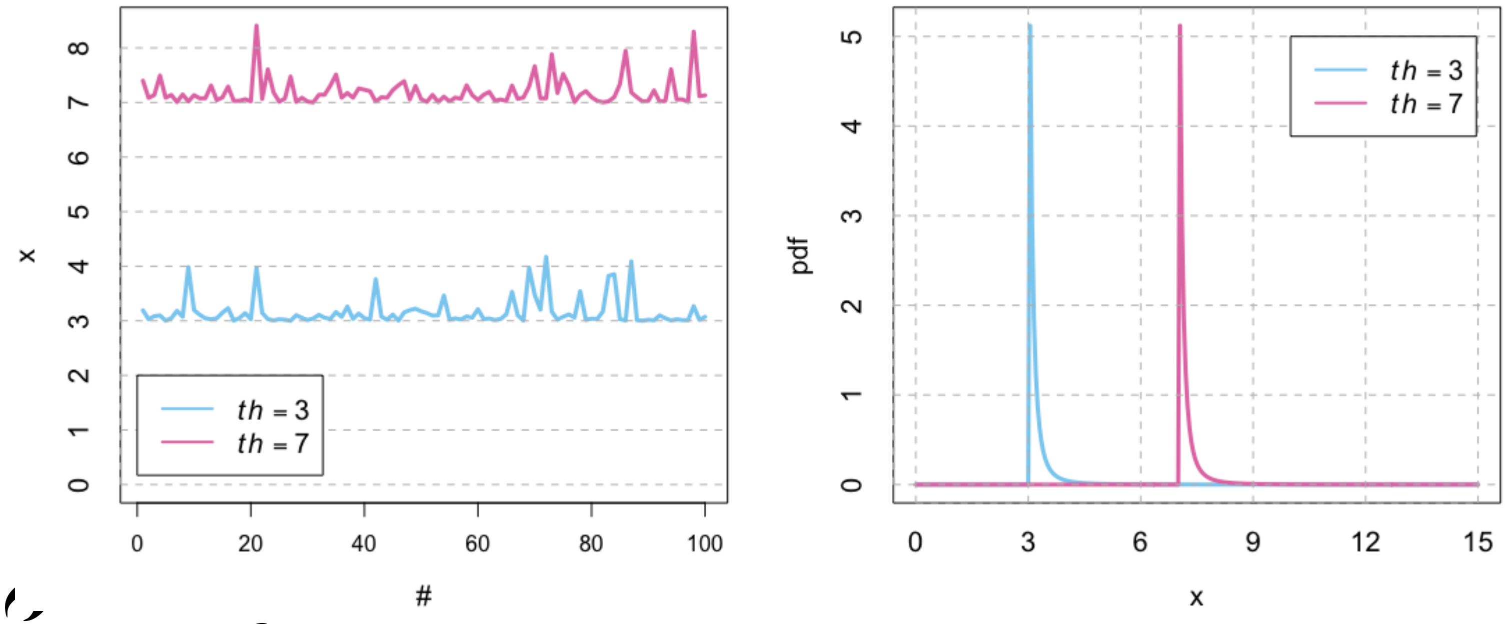
POT and Generalized Pareto Distribution

$$P[X < x | X > th] = \begin{cases} 1 - \left(1 + \frac{\xi(x-th)}{\sigma_{th}}\right)^{-1/\xi} & for\ \xi \neq 0 \\ 1 - exp\left(-\frac{x-th}{\sigma_{th}}\right) & for\ \xi = 0 \end{cases}$$

With parameters threshold (*th*>0), pareto's scale ($\sigma_{th} > 0$) and shape ($-\infty < \xi < \infty$).



Threshold (*th*)

Acts like a location parameter.

POT and Generalized Pareto Distribution

$$P[X < x|X > th] = \begin{cases} 1 - \left(1 + \frac{\xi(x-th)}{\sigma_{th}}\right)^{-1/\xi} & for\ \xi \neq 0 \\ 1 - exp\left(-\frac{x-th}{\sigma_{th}}\right) & for\ \xi = 0 \end{cases}$$

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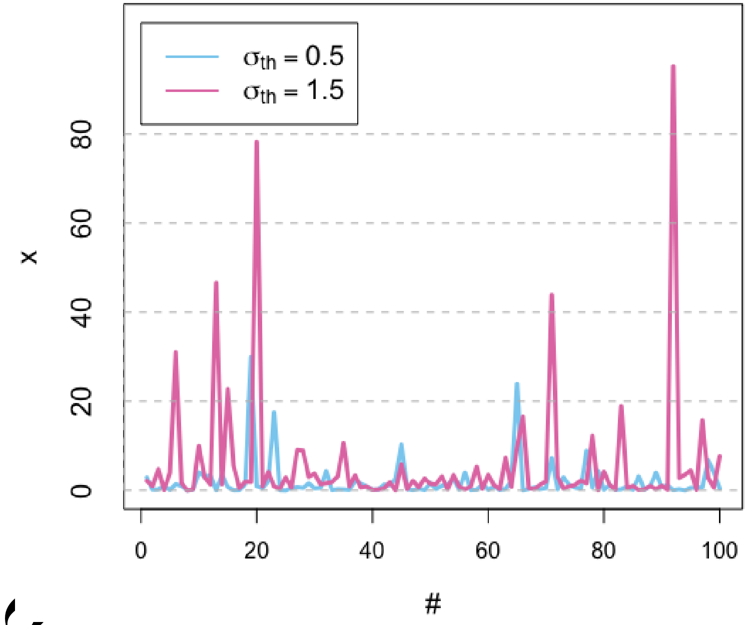

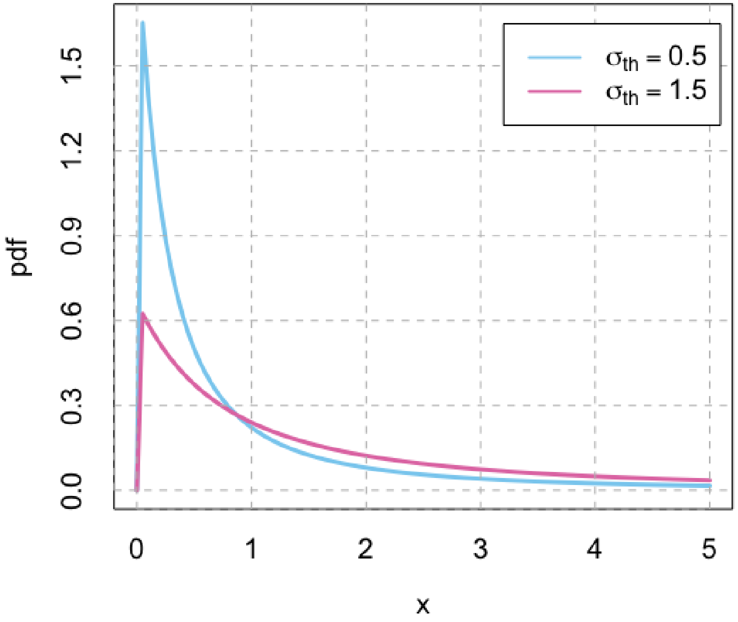

Scale parameter (σ_{th})

Higher $\boldsymbol{\sigma_{th}}$, wider distribution.

POT and Generalized Pareto Distribution

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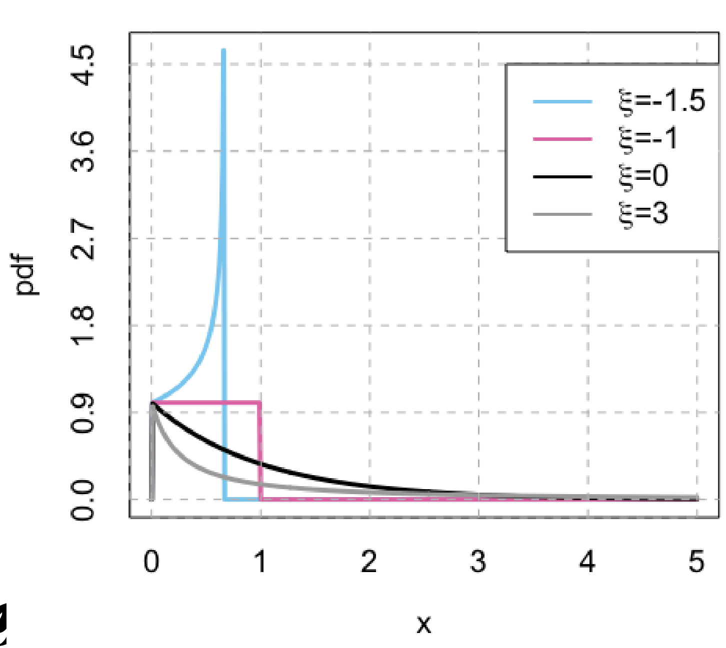

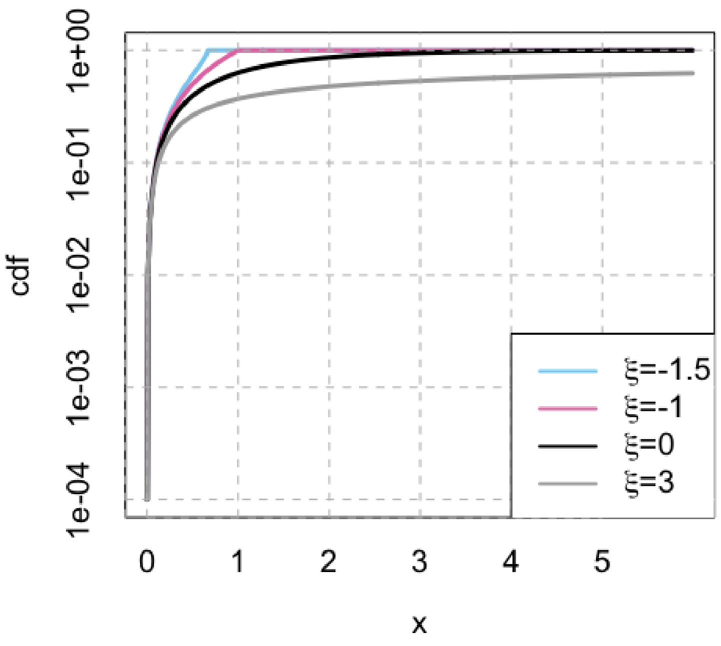

Shape parameter (ξ)

- ξ<0: upper bound
- ξ>0: heavy tail
- ξ=0 & *th* = 0: Exponential
- ξ=-1: Uniform

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Let's talk about the units again…

- **POT** of discharges Q is performed on the observations which last for 5 years. A GPD is fitted to the observations.
- We want to compute the discharge associated with a **return period of 100 years**.

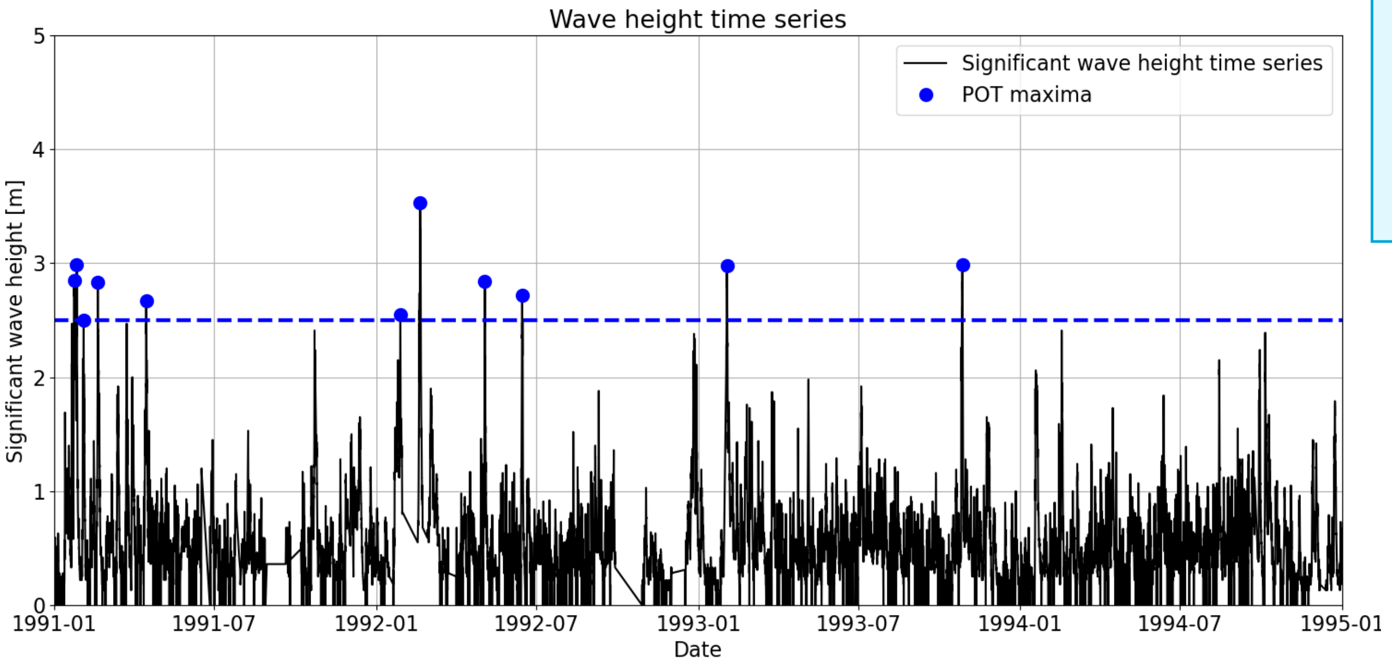
Let’s talk about the units again…

- **POT:** units of the probabilities in the GPD?

- Event-wise probabilities: **not a fixed number in a time block**

- **We use the average number of exceedances per year**

Let's apply it



- **Load: significant wave height (T_R=90 years)**

read observations

th = 2.5
dl = 48 #in hours
excesses = find_peaks(observations, threshold = th, distance = dl) – th

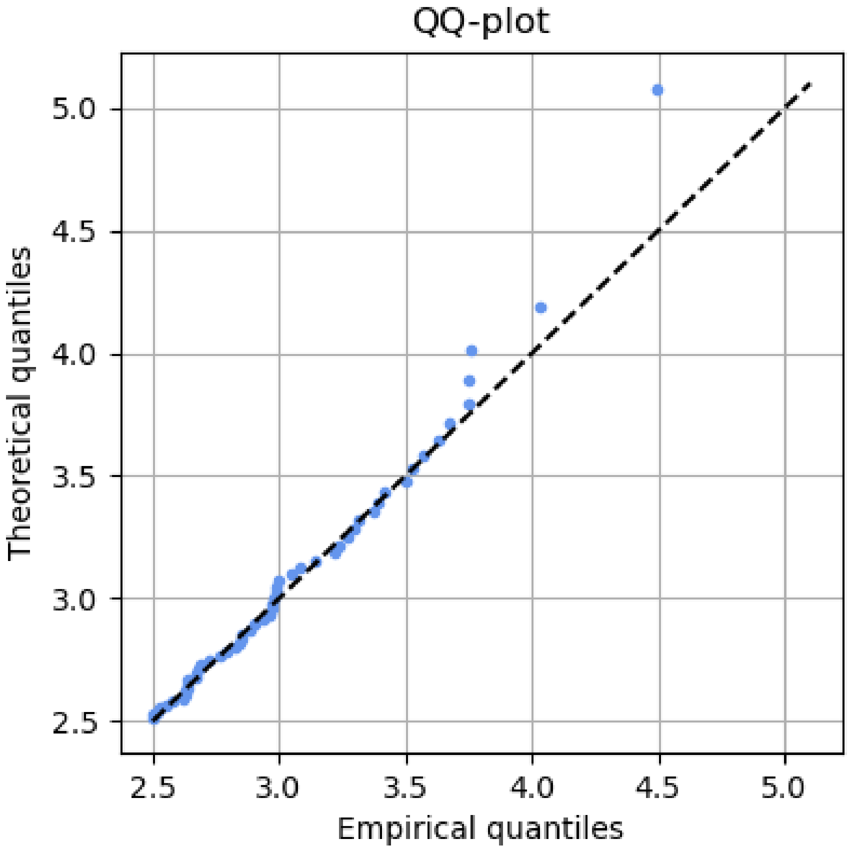
fit GPD(excesses)

check fit (e.g., QQ-plot or Kolmogorov-Smirnov test)

determine lambda

inverse GPD to determine the design event

Let's apply it



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inverse GPD to determine the design event

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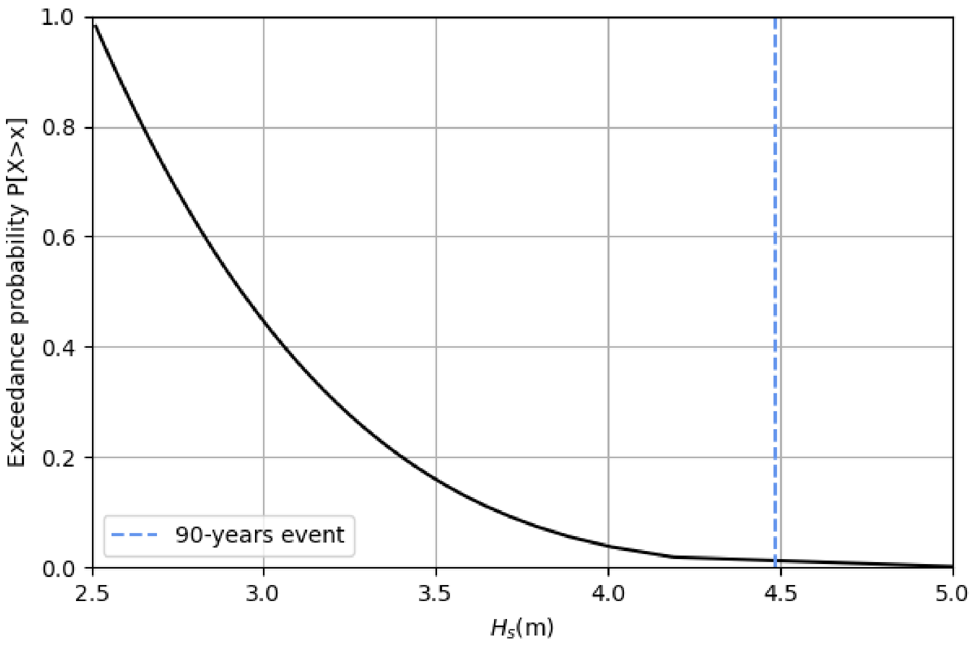
Let's apply it

$$x_N = \begin{cases} th + \frac{\sigma_{th}}{\xi}[(\lambda N)^{\xi} - 1] & for\ \xi \neq 0 \\ th + \sigma_{th} log(\lambda N) & for\ \xi = 0 \end{cases}$$

T_R=90 years
M = 20 years
n_{th} = 54 events

$$\hat{\lambda} = \frac{54}{20} = 2.7$$



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determine lambda

inverse GPD to determine the design event

Learning objectives

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4. Apply techniques to **support the threshold selection**

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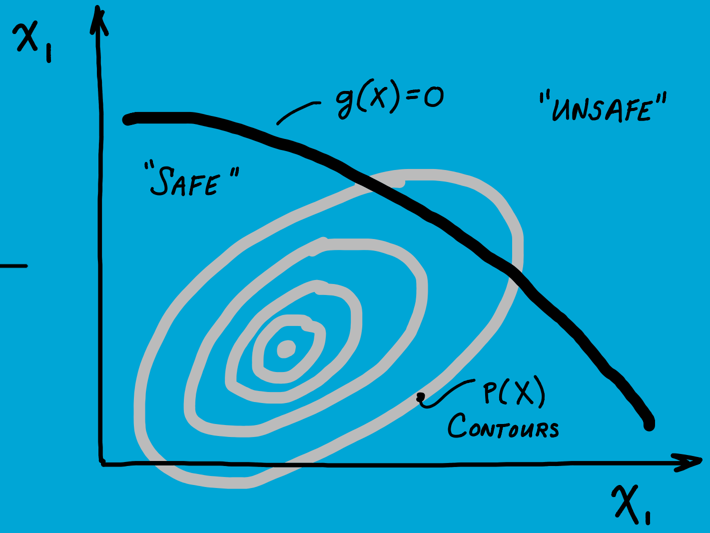
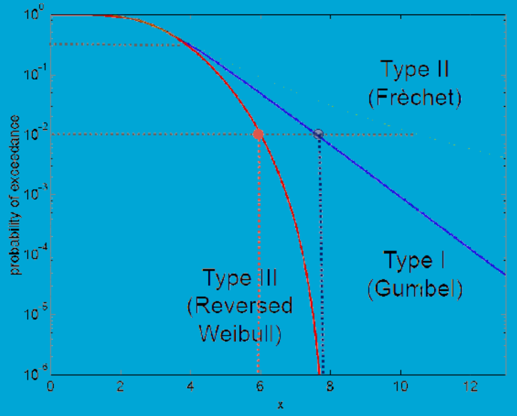
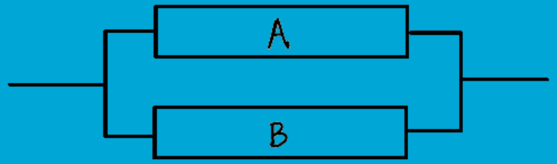
CIEM42X0 Probabilistic Design

Hydraulic and Offshore Structures (HOS) Track

Civil Engineering MSc Program







EVA: Threshold and declustering time selection.

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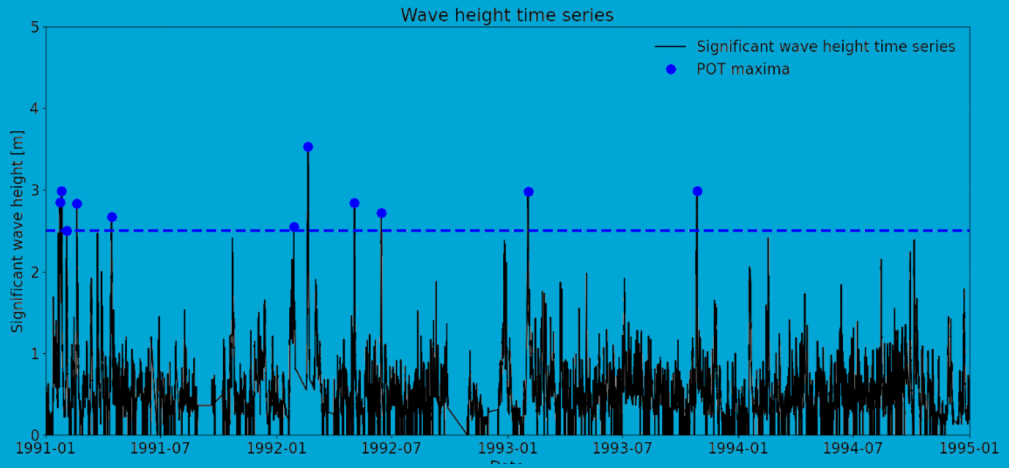

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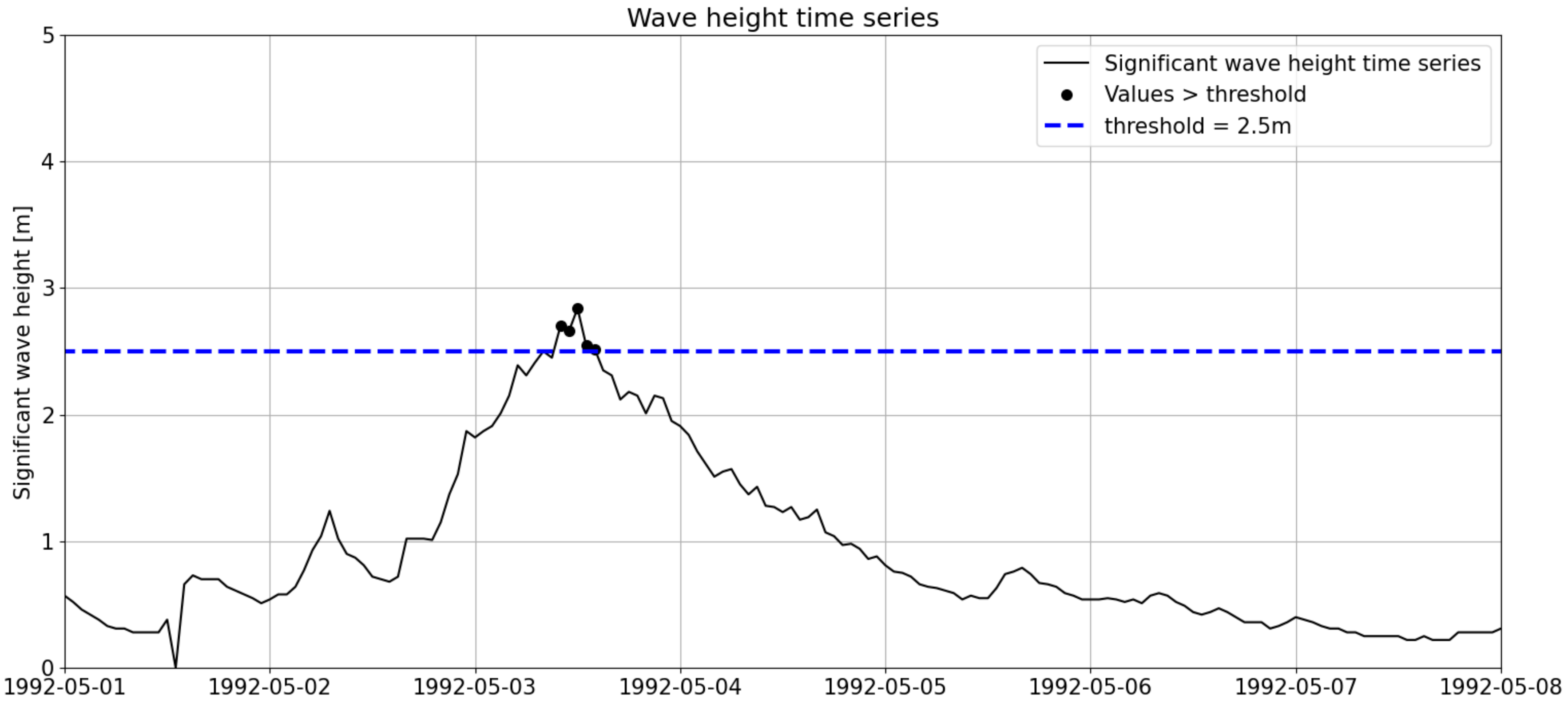
Choosing POT parameters

Basic assumption of EVA: extremes are *iid* ⟹ *th* and *dl* should be chosen so the identified extreme events are independent.

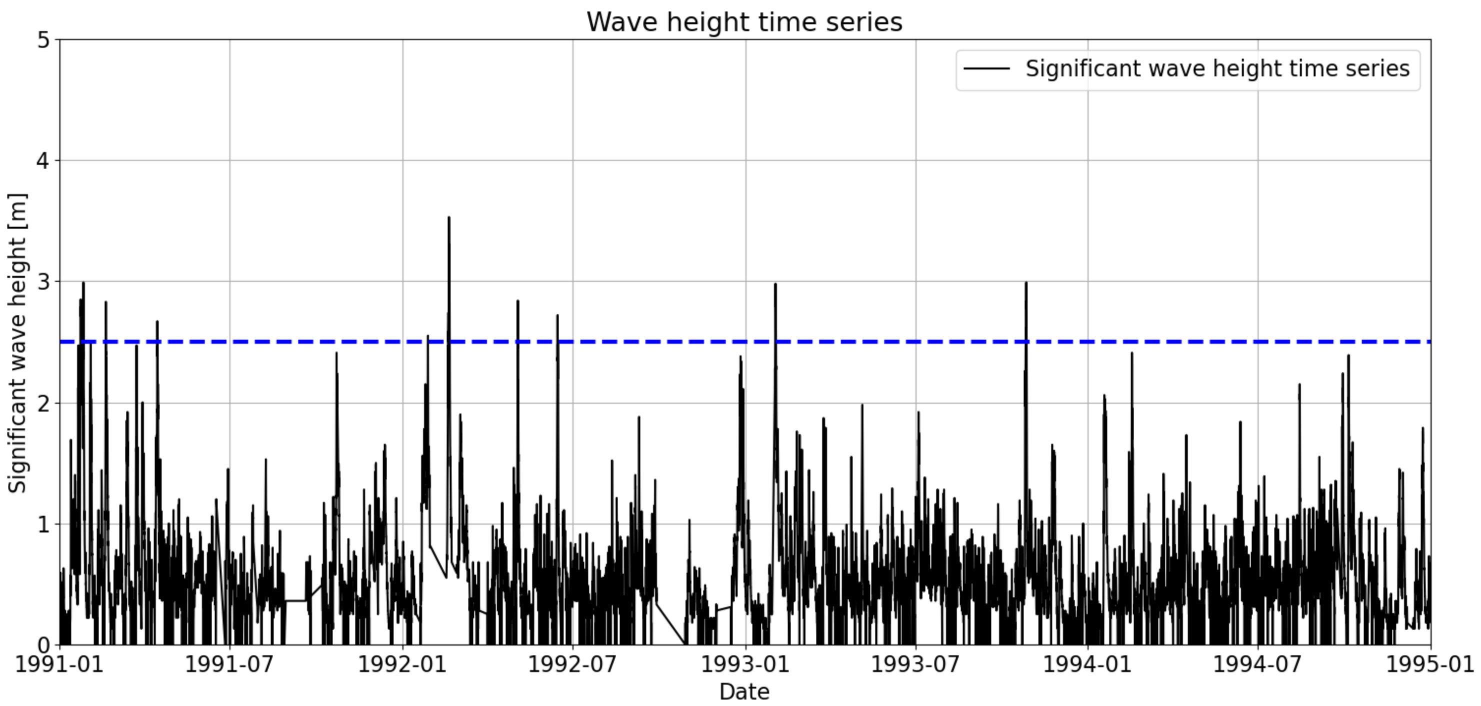


Extremes cluster in time!

If *dl* is big enough, we ensure that extremes do not belong to the same storm.

dl → *th*, physical phenomena (local conditions)

POT and Poisson



- Each hour is a trial ($n \to \infty$)
- Over or below the threshold?
- p_{above} is very small (tail of the distribution)
- Block = 1 year
- Number of excesses over the threshold ~ Poisson

Almost all the techniques to formally select the threshold and declustering time for POT are based on the assumption that the sampled extremes should follow a Poisson distribution.

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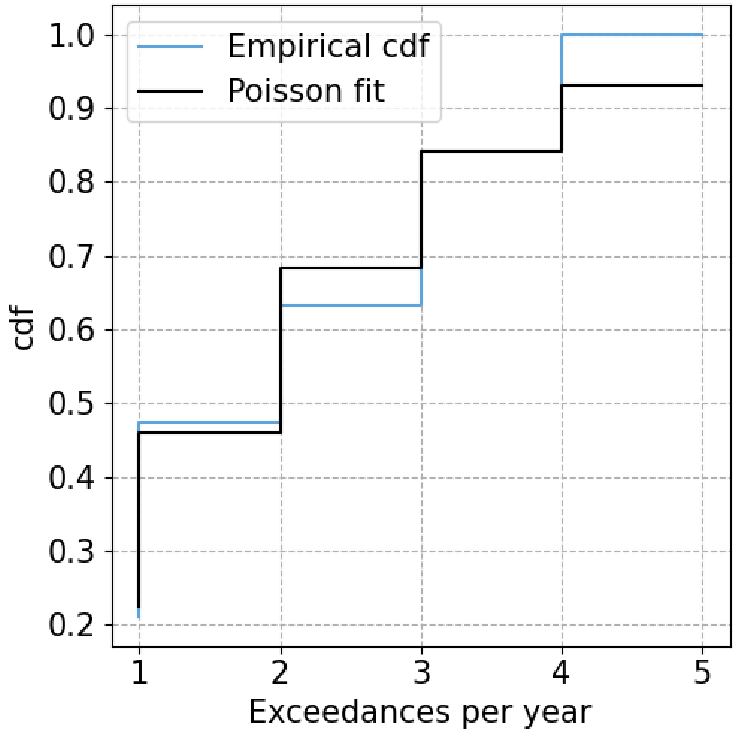
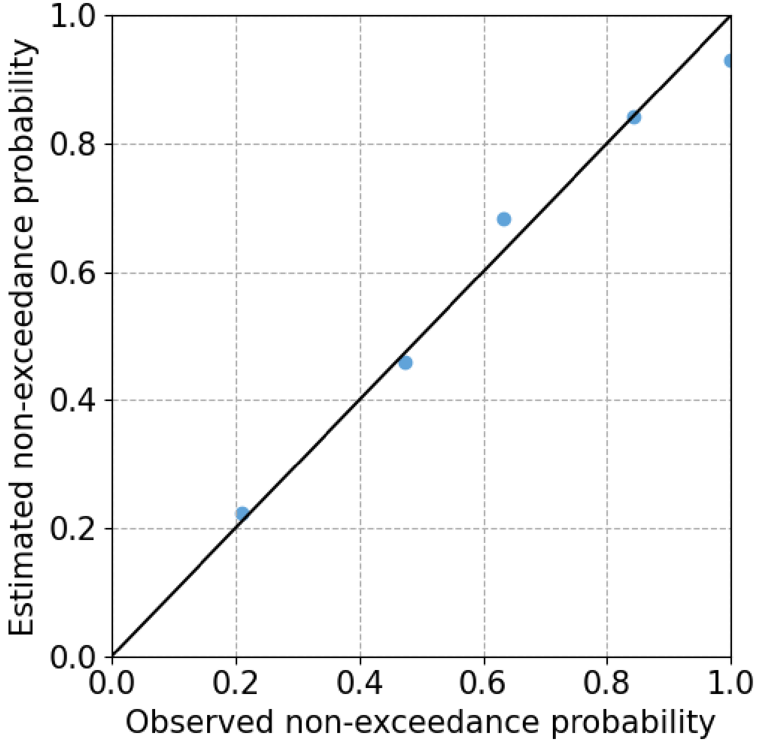
Samples: Poisson

If the number of excesses per year follows a Poisson distribution ⇨ Sampled maxima are independent ✓





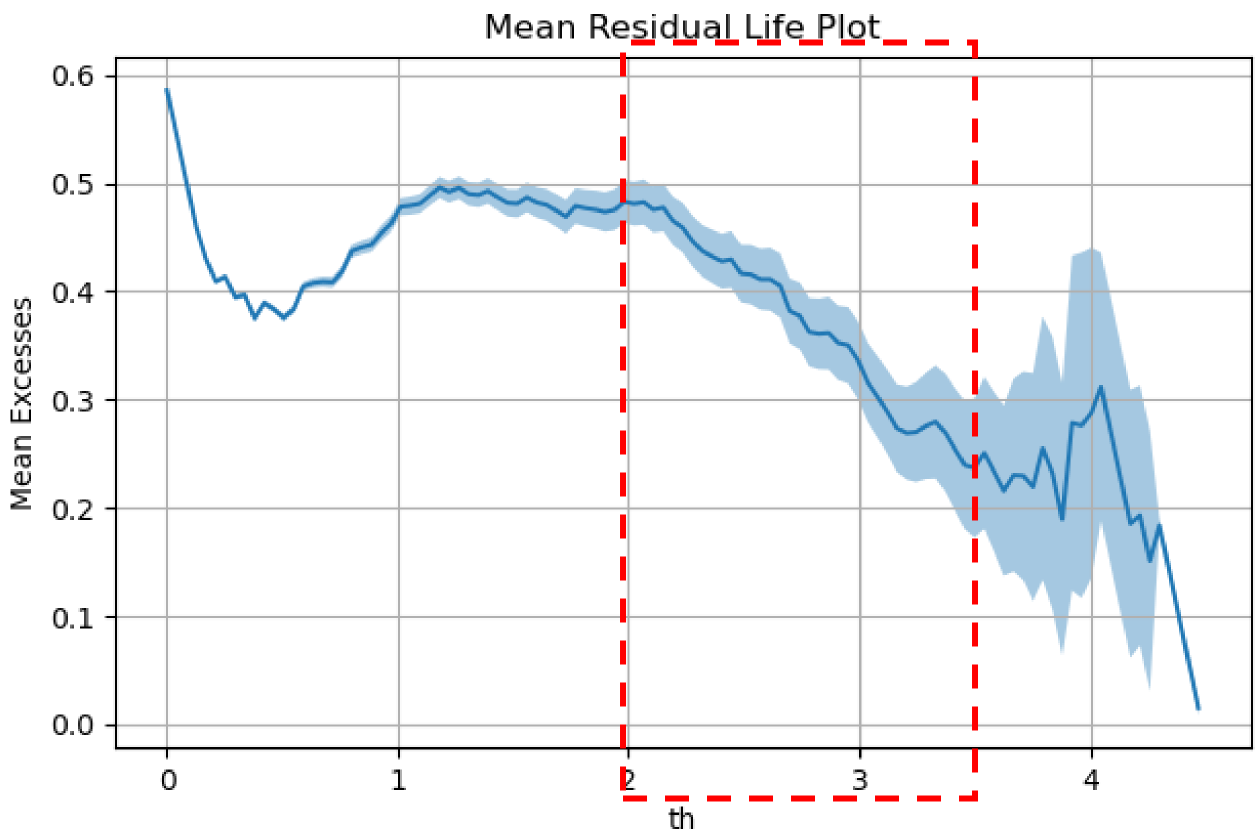
- Compute the number of excesses per year
- Empirical pmf and cdf
- Fit Poisson distribution using Moments

$$E[X] = Var[X] = \lambda$$

- Check the fit
  - Graphically
  - Chi-squared test

Mean Residual Life (MRL) plot

MRL plot presents in the x-axis different values of *th* and, in the y-axis, the mean excess for that value of the *th*. The range of **appropriate threshold** would be that where the **mean excesses follows a linear trend**.



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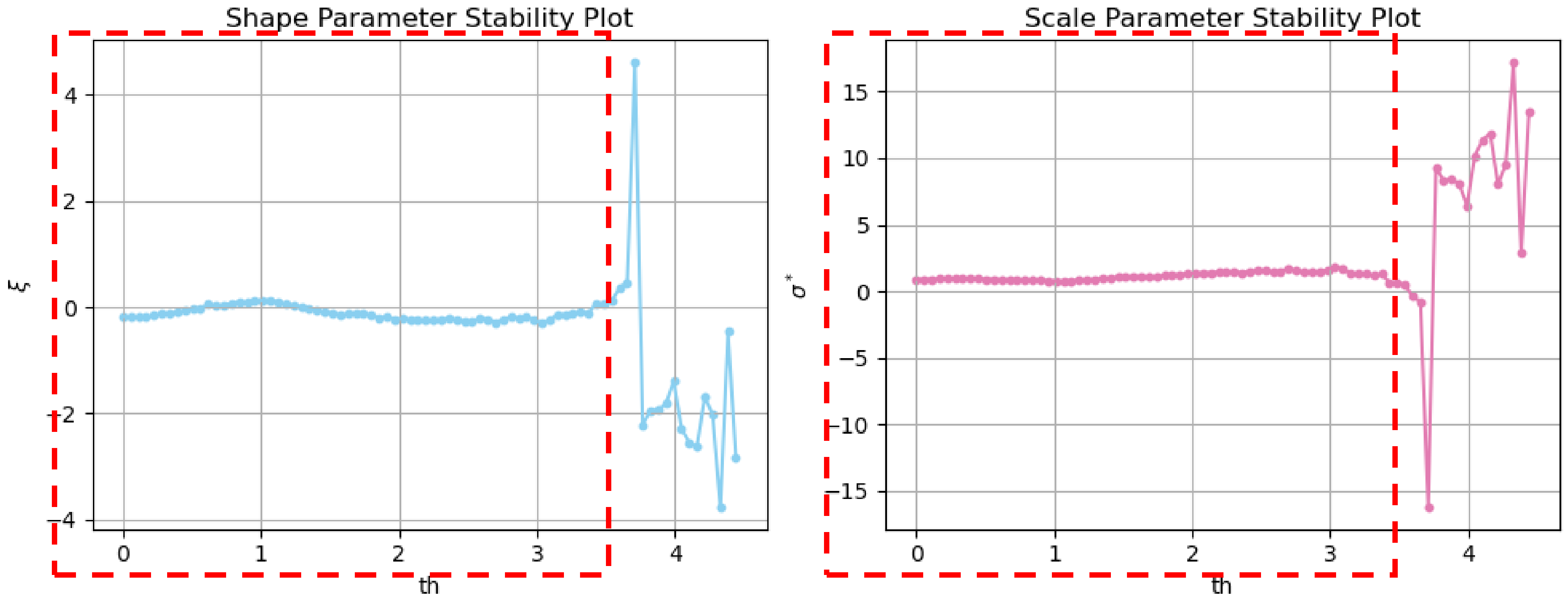
GPD parameter stability plot

GPD distribution is "threshold stable"

If the exceedances over a high threshold (*th0*) a GPD with parameters ξ and σ_{th0}, then for any other threshold (*th>th0)*, the exceedances will also follow a GPD with the same ξ and

$$\sigma_{th} = \sigma_{th0} + \xi(th - th0) \Longrightarrow \sigma^* = \sigma_{th} - \xi th \Longrightarrow \sigma^* = \sigma_{th0} - \xi\, th0$$



Dispersion Index (DI)

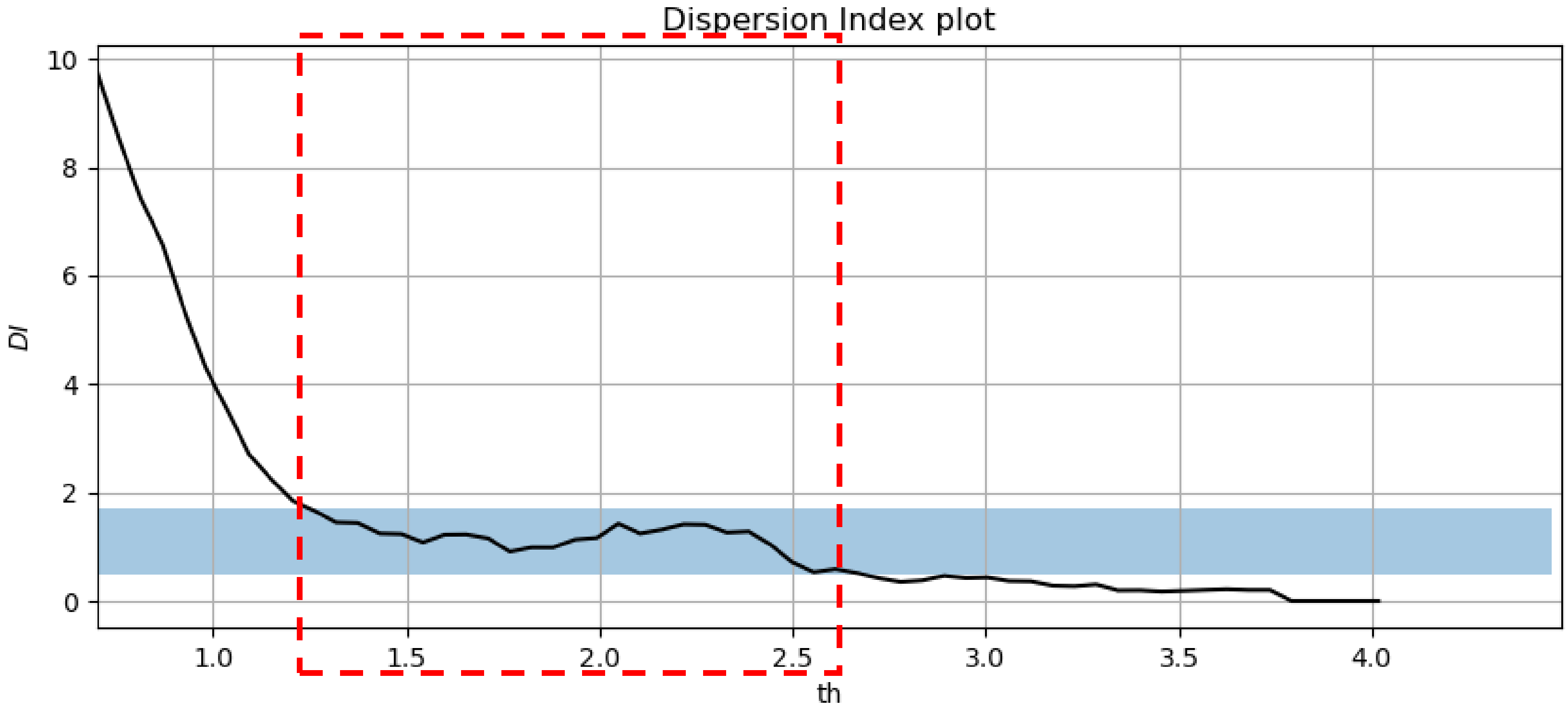
Based on Poisson process

Property of Poisson distribution: $E[X] = Var[X] = \lambda$

Dispersion Index: $DI = \frac{\sigma^2}{\mu} \approx 1$

Confidence interval for DI:

$$\left(\frac{\chi^2_{\alpha/2, M-1}}{(M/1)}, \frac{\chi^2_{1-\alpha/2, M-1}}{(M/1)} \right)$$



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Practicalities… workbook and homework

- **Workbook**: theory with the new methods to select threshold

- **Homework**:
  - Pick one method per person and have the code ready for Friday
  - Hint: there is pseudo code in the book!

Any questions?

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